ALGEBRA COMPREHENSIVE EXAM - JUNE 2011

You have four hours to complete this examination. No references or calculators (not needed) are permitted. There are ten questions, equally weighted. You should attempt all ten and present the work you wish to be graded on the worksheets provided (one sheet per problem). Partial credit will be given. Please try to present your work as legibly as possible. You may use any lemmas or theorems that you can correctly state (other than one you are trying to prove, of course) to establish your arguments. The problems are not presented in any particular order of difficulty, so you should scan the entire list and begin with the ones that seem the easiest to you. Good luck.

All rings are commutative and unital.

1) **State and prove Lagrange's Theorem regarding the possible orders of subgroups of a group**.

2) **If** $\Bbbk[X]$ **is a polynomial ring in** n **indeterminates** $(X = x_1, \cdots, x_n)$ **over the field** $\Bbbk$, **and** $\mathfrak{F} \subset \Bbbk[X]$, **show that** $Var(\mathfrak{F}) = Var(I)$, **where** I **is the ideal generated by** $\mathfrak{F}$. $Var(\mathfrak{F})$ **is the variety induced by** $\mathfrak{F}$.

3) **If** p **is a prime**, **show that every group of order** p^2 **is abelian**.

4) **Show that a group of order** 105 **has a normal subgroup of order** 35.

5) **Show that every non-zero vector space has a Hamel basis**.

6) **Without using Nakayama's Lemma**, **show that if** J **is a proper ideal in a local ring** R, **and** A **is a finitely generated** R-**module such that** $JA = A$, **then** $A = \{0\}$.

7) **State and prove Eisenstein's Criterion for irreducibility over** $\mathbb{Q}$.

8) **Determine the Galois group of** $x^4 - 5x^2 + 6$

9) **Prove or disprove**: **If** G **is a group and** $H, K < G$, **then** $H \cup K < G$ **if and only if either** $H < K$ **or** $K < H$

10) **If** $f : A \to B$ **and** $g : B \to A$ **are** R-**module homomorphisms such that** $gf = 1_A$, **show that** $B \cong Im(f) \oplus Ker(g)$.