Comprehensive Analysis Exam

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Instructions. Answer four of five questions in each part of this exam. Solutions must be written neatly on separate sheet(s) of paper with your name and problem number at the top of each page. Be sure to provide complete and clear reasons for all of your steps (e.g. do not only name a theorem, also show that its hypotheses are satisfied). All problems are equally weighted. You have four (4) hours to submit your solutions.

Part I

In the problems below, all references to "measure", "measurable", "integrable", etc. are with respect to Lebesgue measure on $\mathbb{R}$. The Lebesgue measure of a set A is denoted by $\mu(A)$.

1. Let $\mathcal{C}$ be the Cantor ternary set.

   (a) Describe main steps in the construction of $\mathcal{C}$.

   (b) Prove that $\mathcal{C}$ is measurable, and that its measure is zero.

   (c) Suppose that we repeat the construction of the Cantor set, except that at each stage we remove a fraction d from the center of each of the remaining intervals, where $d < 1/3$. Find the measure of the set $\mathcal{K}$ so constructed.

2. We say that a sequence $\{f_n\}$ of real-valued measurable functions defined on $[0,1]$ is *uniformly integrable* if for every $\epsilon > 0$ there is a $\delta > 0$ such that
$$\sup_n \left| \int_E f_n(x)\,dx \right| < \epsilon$$
for every measurable set $E \subseteq [0,1]$ with $\mu(E) < \delta$.

   (a) Let $\{f_n\}$ be a sequence of *integrable* real-valued functions on $[0,1]$ with $f_n(x) \to f(x)$ almost everywhere with f integrable. Prove that if $\{f_n\}$ is uniformly integrable then
$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 f(x)\,dx.$$

   (b) Give a counterexample to show that the proposition is false if $\{f_n\}$ is not uniformly integrable. In other words, give an example of a sequence $\{f_n\}$ and a function f such that $f_n \to f$ almost everywhere on $[0,1]$ but $\lim\limits_{n\to\infty} \int_0^1 f_n(x)\,dx \neq \int_0^1 f(x)\,dx$.

3. (a) Let $E \subseteq \mathbb{R}$. State the definition of the *outer measure* of E.

   (b) State the definition of what it means for E to be measurable.

   (c) Suppose that $\{f_n\}$ is a sequence of measurable real-valued functions on $\mathbb{R}$. Prove that the set
$$E = \left\{ x \in \mathbb{R} : \lim_{n\to\infty} f_n(x) \text{ exists} \right\}$$
is a measurable set.

4. Recall that a sequence $\{f_n\}$ of measurable functions is said to *converge in measure* to a function f if for every $\varepsilon > 0$, we have $\lim_{n\to\infty} \mu\{x \in \mathbb{R} : |f_n(x) - f(x)| > \epsilon\} = 0$.

   Now, let X be the set of Lebesgue measurable functions on $[0, 1]$, and for $f, g \in X$ define

   $$\rho(f, g) = \int_0^1 \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|}\, dx.$$

   Let $\{f_n\}$ be a sequence of functions in X, and let $f \in X$. Show that $\lim_{n\to\infty} \rho(f_n, f) = 0$ if and only if $\{f_n\}$ converges to f in measure.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a function.

   (a) Describe the stages in the construction of the Lebesgue integral of f.

   (b) Give an example of a function defined on $[0, 1]$ that is Lebesgue integrable but not Riemann integrable.

   (c) Let f be integrable on $[0, 1]$. Prove that $\lim_{k\to\infty} \int_0^1 x^k f(x)\, dx = 0$.

Part II

1. Let (X, d) be a metric space and $A \subseteq X$. Consider the function $d_A : X \to \mathbb{R}$ defined by

$$d_A(x) = \inf\{d(x,a) : a \in A\}, \qquad \text{for } x \in X.$$

   (a) Show that $|d_A(x) - d_A(y)| \le d(x,y)$ for all $x, y \in X$.

   (b) Show that d_A is a continuous function on X.

   (c) Let $\epsilon > 0$. Show that the set $F_\epsilon = \{x \in X : d_A(x) \ge \epsilon\}$ is closed in X.

2. Let f be a continuously differentiable function on $[-\pi, \pi]$ such that

$$f(\pi) = f(-\pi) \quad \text{and} \quad \int_{-\pi}^{\pi} f(x)\,dx = 0.$$

   (a) Show that $\langle f', z^k \rangle = ik \cdot \langle f, z^k \rangle$ for all $k \in \mathbb{Z}$.
   Note: As usual, for $f, g \in L^2[-\pi, \pi]$, we define

$$\langle f, g \rangle \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx.$$

   (b) Prove that $\displaystyle\int_{-\pi}^{\pi} |f(x)|^2\,dx \le \int_{-\pi}^{\pi} |f'(x)|^2\,dx.$

3. Find (with proof) the values of a, b and c in $\mathbb{R}$ that minimize the value of the integral

$$\int_0^\infty |x^3 - a - bx - cx^2|^2 e^{-x}\,dx.$$

4. Let f be a real-valued continuously differentiable function on $\mathbb{R}$ such that

$$\lim_{b \to \infty} b \cdot (f(b))^2 = 0.$$

   Prove that

$$\int_0^\infty (f(x))^2\,dx \le 2\left(\int_0^\infty x^2 \cdot (f(x))^2\,dx\right)^{1/2} \left(\int_0^\infty (f'(x))^2\,dx\right)^{1/2}.$$

5. Find (with proof) the solution $f \in L^2[0,1]$ to the integral equation

$$f(t) = e^t \cos t + \int_0^t f(s)\,ds. \tag{1}$$

   Suggestion: You may use (without proof) the fact that

$$(V^n g)(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} g(s)\,ds, \quad g \in L^2[0,1],$$

   where V denotes the *Volterra operator* on $L^2[0,1]$:

$$(Vg)(t) \stackrel{\text{def}}{=} \int_0^t g(s)\,ds, \quad g \in L^2[0,1].$$