Linear Algebra - Comprehensive Exam. Summer 2011

Instructions

- Complete 8 out of 10 problems.
- Begin each solution on a new page and use additional paper, if necessary.
- Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Notations used
  - $\mathbb{R}$ field of real numbers
  - $\mathbb{C}$ field of complex numbers
  - $dim(\mathbb{V})$ dimension of a vector Space
  - $\mathbb{F}$ either $\mathbb{C}$ or $\mathbb{R}$.
  - $\mathbb{F}^n$ set of n-tuples $(x_1, \ldots, x_n),\ x_i \in \mathbb{F}$
  - $\mathcal{L}(\mathbb{V})$ set of linear operators $T : \mathbb{V} \longmapsto \mathbb{V}$
  - $\mathcal{L}(\mathbb{V}, \mathbb{W})$ set of linear transformations $T : \mathbb{V} \longmapsto \mathbb{W}$
  - $\mathcal{M}(m, n, \mathbb{F})$ vector space of $m \times n$ matrices with entries from $\mathbb{F}$.
  - $\mathbb{P}(\mathbb{F})$ set of polynomials over $\mathbb{F}$
  - $\mathbb{P}_m(\mathbb{F})$ set of polynomials of degree at most m over $\mathbb{F}$
  - $span(v_1, \ldots, v_n)$ span of a list of vectors
  - $\mathbb{V} \oplus \mathbb{W}$ direct sum of $\mathbb{V}$ and $\mathbb{W}$
  - $\mathbb{V}(\mathbb{F})$ vector space over $\mathbb{F}$
  - A_{ij} $(i, j)^{th}$ element of matrix A
  - $\emptyset$ null set
- We reserve the right to deduct points for matters of unclear or disproportionally cumbersome presentation.

1. Consider a linear transformation $T \in \mathcal{L}(\mathcal{M}(2,2,\mathbb{R}),\mathbb{R})$ given by

$$T(A) = \sum_{i=1}^{2} A_{ii}, \quad A \in \mathcal{M}(2,2,\mathbb{R})$$

   (a) Find the matrix of transformation T with respect to standard bases
   (b) Find a basis and the dimension of null space of T

   Note that the standard basis for $\mathcal{M}(2,2,\mathbb{R})$ is
   $\beta = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$

2. Consider the following subsets of $\mathbb{R}^3$
   $\mathbb{W}_1 = \{x \in \mathbb{R}^3 | x = (0, x_2, x_3)\}$, $\mathbb{W}_2 = \{x \in \mathbb{R}^3 | x = (x_1, 0, x_3)\}$, $\mathbb{W}_3 = \{x \in \mathbb{R}^3 | x = (x_1, x_2, 0)\}$
   (a) Prove whether each of the following sets are subspaces of $\mathbb{R}^3$.
      i. $\mathbb{W}_1$
      ii. $\mathbb{W}_1 \cap \mathbb{W}_2$
      iii. $\mathbb{W}_1 \cup \mathbb{W}_2$
      iv. $\mathbb{W}_1 \cap \mathbb{W}_2 \cap \mathbb{W}_3$
   (b) Show that $\mathbb{R}^3 = \mathbb{W}_{12} \oplus \mathbb{W}_{23} \oplus \mathbb{W}_{31}$, where $\mathbb{W}_{ij} = \mathbb{W}_i \cap \mathbb{W}_j$

3. $A = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 6 & 2 \end{bmatrix}$

   (a) Determine the eigenvalues of A and a basis for each eigenspace.
   (b) Find an invertible matrix R such that $R^{-1}AR$ is a diagonal matrix
   (c) Use the eigenstructure of A to determine the vector
   $A^{53} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

4. (a) Let $t \in \mathbb{R}$ such that t is not an integer multiple of π. For the matrix $A = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$ prove there does not exist a **real** valued matrix B such that BAB^{-1} is a diagonal matrix.
   (b) Do the same for matrix $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ where $a \in \mathbb{R} \backslash \{0\}$

5. Let $\mathbb{W}_1$ and $\mathbb{W}_2$ be subspaces of vector space $\mathbb{V}$ such that $\mathbb{V} = \mathbb{W}_1 \oplus \mathbb{W}_2$. If β_1 and β_2 are bases for $\mathbb{W}_1$ and $\mathbb{W}_2$, respectively, show that $\beta_1 \cap \beta_2 = \emptyset$ and $\beta_1 \cup \beta_2$ is a base for $\mathbb{V}$

6. Prove the following theorem. Let $\mathbb{U}_1, \ldots, \mathbb{U}_n$ be subspaces of $\mathbb{V}$. Then $\mathbb{V} = \mathbb{U}_1 \oplus \cdots \oplus \mathbb{U}_n$ if and only if both of the following are true:
   (a) $\mathbb{V} = \mathbb{U}_1 + \cdots + \mathbb{U}_n$
   (b) The only way to write 0 as a sum $u_1 + \cdots + u_n$, with each $u_j \in \mathbb{U}_j$, is if $u_j = 0$ for all the j.

7. Suppose $(v_1, ..., v_n)$ is a basis of $\mathbb{V}$. Prove that function $T : \mathbb{V} \to \mathcal{M}(n, 1, \mathbb{F})$ defined by

$$Tv = M(v)$$

   is linear and invertible.
   Note: $M(v)$ is the matrix of $v \in \mathbb{V}$ with respect to the basis $(v_1, ..., v_n)$.

8. Prove or give a counterexample: If $\mathbb{U}$ is a subspace of $\mathbb{V}$ that is invariant under every linear operator on $\mathbb{V}$, then $\mathbb{U} = \{0\}$ or $\mathbb{U} = \mathbb{V}$.

9. Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces and $T \in \mathcal{L}(\mathbb{V}, \mathbb{W})$. Suppose $\mathbb{V}$ is finite dimensional. Show that
$$dim(\mathbb{V}) = dim(range(T)) + dim(null(T))$$

10. Let $\mathbb{V}$ be a finite dimensional vector space, and let $\mathbb{U}$ be a subspace of $\mathbb{V}$. Prove that there is a subspace $\mathbb{W}$ of $\mathbb{V}$ such that $\mathbb{V} = \mathbb{U} \oplus \mathbb{W}$.