Comprehensive Analysis Exam

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Saturday, January 14, 2012

Instructions. Answer four of five questions in each part of this exam. Solutions must be written neatly on separate sheet(s) of paper with your name and problem number at the top of each page. Be sure to provide complete and clear reasons for all of your steps (e.g. do not only name a theorem, also show that its hypotheses are satisfied). All problems are equally weighted. You have four (4) hours to submit your solutions.

Part I

In the problems below, all references to "measure", "measurable", "integrable", etc. are with respect to Lebesgue measure on $\mathbb{R}$. The Lebesgue measure of a set A is denoted by $\mu(A)$.

1. Let $\mathcal{C}$ be the Cantor ternary set.
   (a) Describe main steps in the construction of $\mathcal{C}$.
   (b) Prove that $\mathcal{C}$ is measurable, and that its measure is zero.

2. Let f be a measurable function defined on an interval $[a, b]$.
   (a) State what it means (i.e. give the definition) to say that f is *absolutely continuous* on $[a, b]$.
   (b) Suppose that f is differentiable on $[a, b]$, and that f' is bounded on $[a, b]$. Prove that f is absolutely continuous on $[a, b]$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function.
   (a) Describe the stages in the construction of the Lebesgue integral of f.
   (b) Give an example of a function defined on $[0, 1]$ that is Lebesgue integrable but not Riemann integrable.
   (c) Let f be integrable on $[0, 1]$. Prove that $\lim_{k\to\infty} \int_0^1 x^k f(x)\, dx = 0$.

4. (a) Let $E \subseteq \mathbb{R}$. State the definition of the *outer measure* of E.
   (b) State the definition of what it means for E to be Lebesgue measurable.
   (c) Suppose that $\{f_n\}$ is a sequence of (Lebesgue) measurable real-valued functions on $\mathbb{R}$. Prove that the set
   $$E = \left\{x \in \mathbb{R} : \lim_{n\to\infty} f_n(x) \text{ exists}\right\}$$
   is a measurable set.

5. Recall that a sequence $\{f_n\}$ of measurable functions is said to *converge in measure* to a function f if for every $\varepsilon > 0$, we have $\lim_{n\to\infty} \mu\{x \in \mathbb{R} : |f_n(x) - f(x)| > \epsilon\} = 0$.
   (a) Give an example of a sequence $\{f_n\}$ of measurable functions and a measurable function f, all defined on $[0, 1]$, such that $f_n \to f$ in measure but f_n does not converge pointwise.
   (b) Give an example of a sequence $\{f_n\}$ of measurable functions and a measurable function f, all defined on $[0, \infty)$, such that $f_n \to f$ pointwise but f_n does not converge to f in measure.

Part II

Throughout, $C(K)$ denotes the collection of continuous functions real-valued functions on K equipped with the supremum norm $\|f\|_{C(K)} = \sup\{|f(x)| : x \in K\}$.

1. (a) State the definition of a Cauchy sequence in a metric space (X, d).

   (b) For each $n \in \mathbb{N}$, let $g_n(x) = \dfrac{nx}{1+nx^2}$ for $x \in [0,1]$.
   Is the sequence of functions $(g_n)_{n\in\mathbb{N}}$ a Cauchy sequence in $C([0,1])$? Prove that your assertion is correct.

2. Let $\mathcal{H}$ be a Hilbert space with an orthonormal set $(e_n)_{n\in\mathbb{N}}$. Show that if $x \in \mathcal{H}$, then

$$\lim_{n\to\infty} \langle x, e_n \rangle = 0.$$

3. Let K be a compact subset of $\mathbb{R}$.

   (a) State the Arzelà-Ascoli Theorem for $C(K)$

   (b) For each $n \in \mathbb{N}$, let $f_n(x) = x^n$ for $x \in [0,1]$. Prove that the sequence of functions $(f_n)_{n\in\mathbb{N}}$ is *not* equicontinuous.

   (c) Prove that the closed unit ball $\mathcal{F} = \{f \in C([0,1]) : \|f\|_{C[0,1]} \leq 1\}$ is *not* a compact subset of $C([0,1])$.

4. For $f, g \in L^2[-\pi,\pi]$, recall that $\langle f, g \rangle = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx$.

   Let $h(x) = x$ for $x \in [-\pi, \pi]$.

   (a) Compute the Fourier coefficients of h in $L^2[-\pi,\pi]$.

   (b) Use Plancherel's theorem and the answer obtained in part 4a to show that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

5. Let $\Lambda : C[0,1] \to \mathbb{C}$ be defined by $\Lambda(f) = \displaystyle\int_0^1 t \cdot f(t)\,dt$.

   (a) Show that Λ defines a linear functional on $C[0,1]$.

   (b) Compute the exact value of $\|\Lambda\|$.