Linear Algebra - Comprehensive Exam. Spring 2012

Instructions

- Complete 8 out of 10 problems.
- Begin each solution on a new page and use additional paper, if necessary.
- Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- Calculators are only permitted for basic arithmetics and computation of *rref*
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Notations used
  - $\mathbb{R}$ field of real numbers
  - $\mathbb{C}$ field of complex numbers
  - $dim(\mathbb{V})$ dimension of a vector Space
  - $\mathbb{F}$ either $\mathbb{C}$ or $\mathbb{R}$.
  - $\mathbb{F}^n$ set of n-tuples $(x_1, \ldots, x_n),\ x_i \in \mathbb{F}$
  - $\mathcal{L}(\mathbb{V})$ set of linear operators $T : \mathbb{V} \longmapsto \mathbb{V}$
  - $\mathcal{L}(\mathbb{V}, \mathbb{W})$ set of linear transformations $T : \mathbb{V} \longmapsto \mathbb{W}$
  - $\mathcal{M}(m, n, \mathbb{F})$ vector space of $m \times n$ matrices with entries from $\mathbb{F}$.
  - $\mathbb{P}(\mathbb{F})$ set of polynomials over $\mathbb{F}$
  - $\mathbb{P}_m(\mathbb{F})$ set of polynomials of degree at most m over $\mathbb{F}$
  - $span(v_1, \ldots, v_n)$ span of a list of vectors
  - $\mathbb{V} \oplus \mathbb{W}$ direct sum of $\mathbb{V}$ and $\mathbb{W}$
  - $\mathbb{V}(\mathbb{F})$ vector space over $\mathbb{F}$
  - A_{ij} $(i, j)^{th}$ element of matrix A
  - $\emptyset$ null set
- We reserve the right to deduct points for matters of unclear or cumbersome presentation.

1. Let $S = \{x_1, x_2, \ldots, x_n\} \in \mathbb{R}$. Let $W = \{p \in \mathbb{P}(\mathbb{R}) \mid p(x_i) = 0 \text{ for all } 1 \leq i \leq n\}$ be the set of those polynomials with real coefficients which are 0 at every point in S. Is W a subspace of $\mathbb{P}(\mathbb{R})$? If so prove it, if not, explain why not.

2. Suppose that $\mathbb{V}$ and $\mathbb{W}$ are both finite dimensional subspaces. Prove that there exists a surjective linear map from $\mathbb{V}$ onto $\mathbb{W}$ **iff** $dim(\mathbb{W}) \leq dim(\mathbb{V})$

3. Prove or give a counterexample: If $\mathbb{U}$ is a subspace of $\mathbb{V}$ that is invariant under every linear operator on $\mathbb{V}$, then $\mathbb{U} = \{0\}$ or $\mathbb{U} = \mathbb{V}$.

4. Suppose $T \in \mathcal{L}(\mathbb{V})$ is such that every vector in $\mathbb{V}$ is an eigenvector of T. Prove that T is a scalar multiple of the identity operator.

5. Prove that if $p,\ q \in \mathbb{P}(\mathbb{F})$, with $p \neq 0$, then there exist unique polynomials $s,\ r \in \mathbb{P}(\mathbb{F})$ such that
$$q = sp + r$$
and $deg\ r\ <\ deg\ p$.

6. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation with the property that $T \circ T \circ T = 0$. We will refer $T \circ T \circ T$ as T^3 in the rest of the problem.
   (a) Suppose that $x \in \mathbb{R}^3$ is such that $T^2(x) = T(T(x)) \neq 0$. If $z = cT^2(x)$, then find $T(z)$. If $y = c_1T(x) + c_2T^2(x)$ find $T^2(y)$.
   (b) If $b_1 = x$, $b_2 = T(x)$ and $b_3 = T^2(x)$. Show that b_1 is not a linear combination of b_2 and b_3.

7. Suppose $A \in \mathcal{M}(m, n, \mathbb{R})$ and $B \in \mathcal{M}(n, s, \mathbb{R})$ and both matrices have full column rank (columns are linearly independent). Suppose furthermore that the range of AB is equal to the range of A. What relations (e.g. $<, >, \leq, \geq, =$) must hold between the integers m, n and s? Explain the reasoning and then give an example that illustrate your conclusion.

8. Consider the linear transformation $T(f) = 3f'' - 2f'$ from $\mathbb{P}_2$ to $\mathbb{P}_2$ and let $\beta = (1, t, t^2)$ be a basis for $\mathbb{P}_2$.
   (a) Find the β-**matrix of linear transformation** T.
   (b) Determine whether T is an isomorphism.
   (c) If T is not an isomorphism, find bases for the *kernel* and *image* of transformation T.

9. Prove the following theorem. Let $\mathbb{V}$ be a vector space with subspaces $\mathbb{U}$ and $\mathbb{W}$. Then $\mathbb{V} = \mathbb{U} \oplus \mathbb{W}$ **iff** $\mathbb{V} = \mathbb{U} + \mathbb{W}$ and $\mathbb{U} \cap \mathbb{W} = \{0\}$.

10. Show that if S_1 and S_2 are arbitrary subsets of a vector space $\mathbb{V}$, then $span(S_1 \cup S_2) = span(S_1) + span(S_2)$