ALGEBRA COMPREHENSIVE EXAM - AUGUST 2012

You have three hours to complete this examination. No references or calculators (not needed) are permitted. Please attempt to **answer all questions**, which are equally weighted. You should present the work you wish to be graded on the scratch sheets provided (one sheet per problem). Partial credit will be given. Please try to present your work as legibly as possible. You may use any lemmas or theorems that you can correctly state (other than one you are trying to prove, of course) to establish your arguments. The problems are not presented in any particular order of difficulty, so you should scan the entire list and begin with the ones that seem the easiest to you. Good luck.

1) Prove or disprove: There exists a group of order 56 that is simple.

2) Suppose G is an abelian group of order 720. Find all isomorphism classes of G.

3) Suppose $H \leq G$. Show that $H \trianglelefteq G$ if and only if the commutator $[H,G] \leq H$.

4) Show that $f(x) = x^4 + 1$ is irreducible over $\mathbb{Q}$.

5) Prove or disprove: A finite field may be algebraically closed.

6) Let X be any nonvoid set and define addition on the power set $\wp(X)$ by symmetric difference and multiplication by intersection (i.e. for $A, B \subset X$, $A \oplus B = (A - B) \cup (B - A)$ and $A \otimes B = A \cap B$). Show that $\langle \wp(X), \oplus, \otimes \rangle$ is a ring with the property that for all $A \subset X$, $A \otimes A = A$ (and hence it is a boolean ring).