Comprehensive Analysis Exam

Department of Mathematics
Florida Gulf Coast University
Saturday, August 25, 2012

Instructions. No references are permitted during the exam. Solutions must be written legibly and neatly on separate sheet(s) of paper with your name and problem number at the top of each page. Be sure to provide complete and clear reasons for all of your steps (e.g. no statement is "clear" or "obvious" unless it's a definition or given as an assumption, and do not only name a theorem, also show that its hypotheses are satisfied). All problems are equally weighted. You have three (3) hours to submit your solutions.

In the problems below, all references to "measure", "measurable", "integrable", etc. are with respect to Lebesgue measure on $\mathbb{R}^d$. The Lebesgue measure of a set A is denoted by $\boldsymbol{m}(A)$.

1. Give an example of a compact set K such that K', the set of limit points of K, is infinite and countable. Be sure to explain why the set you provide has *all* of the desired properties.

2. Let $\mathcal{C}$ be the Cantor ternary set.

   (a) Describe main steps in the construction of $\mathcal{C}$ and show that $\mathcal{C}$ is a closed set.

   (b) Prove that $\mathcal{C}$ is a perfect set.
   Remark. A closed set E is said to be *perfect* if every point of E is a limit point of E.

3. Let $g : [0,1] \to \mathbb{R}$ be continuous. Show that the function $h : [0,1] \to \mathbb{R}$ defined by

$$h(x) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{[g(x)]^n}{(1+|g(x)|)^n}$$

   is continuous on $[0,1]$.

4. Evaluate the limit $\displaystyle\lim_{n\to\infty} \int_1^n e^{x/n} \frac{1}{x^2}\,dx$ and prove that your answer is correct, including proving that the limit exists. You can assume basic calculus results.

5. Let $E \subseteq \mathbb{R}^d$ be such that $\boldsymbol{m}^*(E) < \infty$. Define $-E \stackrel{\text{def}}{=} \{-p : p \in E\}$.

   (a) Show that $\boldsymbol{m}^*(-E) = \boldsymbol{m}^*(E)$.

   (b) Use part 5a to prove that if E is Lebesgue measurable, then $-E$ is also Lebesgue measurable.

6. Let $\mathcal{K}$ be the measurable subset of $[0,1]$ constructed in an analogous manner as the Cantor ternary set $\mathcal{C}$ except that each of the open intervals removed at the nth stage has length $2/15^n$. Let $\mathcal{O}_n$ denote the union of the 2^{n-1} open disjoint intervals removed at the nth stage.

   (a) Show $\mathcal{K}$ is a Lebesgue measurable set and compute $\boldsymbol{m}(\mathcal{K})$.

   (b) Consider the function $f : [0,1] \to [0,\infty]$ defined by setting $f(x) = \frac{13}{3}$ for $x \in \mathcal{K}$ and $f(x) = 3^n$ when $x \in \mathcal{O}_n$ for some $n \in \mathbb{N}$. Show that f is measurable and evaluate $\displaystyle\int_{[0,1]} f\,d\boldsymbol{m}$.