Linear Algebra - Comprehensive Exam. Summer 2012

Instructions

- Answer all questions.
- Begin each solution on a new page and use additional paper, if necessary.
- Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- Calculators are not allowed.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Notations used
  - $\mathbb{R}$ field of real numbers
  - $\mathbb{C}$ field of complex numbers
  - $dim(\mathbb{V})$ dimension of a vector Space
  - $\mathbb{F}$ either $\mathbb{C}$ or $\mathbb{R}$.
  - $\mathbb{F}^n$ set of n-tuples $(x_1, \ldots, x_n),\ x_i \in \mathbb{F}$
  - $\mathcal{L}(\mathbb{V})$ set of linear operators $T : \mathbb{V} \longmapsto \mathbb{V}$
  - $\mathcal{L}(\mathbb{V}, \mathbb{W})$ set of linear transformations $T : \mathbb{V} \longmapsto \mathbb{W}$
  - $\mathcal{M}(m, n, \mathbb{F})$ vector space of $m \times n$ matrices with entries from $\mathbb{F}$.
  - $\mathbb{P}(\mathbb{F})$ set of polynomials over $\mathbb{F}$
  - $\mathbb{P}_m(\mathbb{F})$ set of polynomials of degree at most m over $\mathbb{F}$
  - $span(v_1, \ldots, v_n)$ span of a list of vectors
  - $\mathbb{V} \oplus \mathbb{W}$ direct sum of $\mathbb{V}$ and $\mathbb{W}$
  - $\mathbb{V}(\mathbb{F})$ vector space over $\mathbb{F}$
  - A_{ij} $(i, j)^{th}$ element of matrix A
  - $\emptyset$ null set
  - $I_{\mathbb{V}}$ Identity transformation on $\mathbb{V}$.
  - $< u, v >$ Inner product of vector u and v.
- We reserve the right to deduct points for matters of unclear or disproportionally cumbersome presentation.

1. Let $\mathbb{U}_1, \mathbb{U}_2, \ldots, \mathbb{U}_k$ be subspaces of a vector space $\mathbb{V}$. Set $\mathbb{W}_1 = \mathbb{U}_2 + \mathbb{U}_3 + \cdots + \mathbb{U}_k$. For $1 < i < k$, set $\mathbb{W}_i = \mathbb{U}_1 + \cdots + \mathbb{U}_{i-1} + \mathbb{U}_{i+1} + \cdots + \mathbb{U}_k$ and $\mathbb{W}_k = \mathbb{U}_1 + \mathbb{U}_2 + \cdots + \mathbb{U}_{k-1}$. Prove that $\mathbb{V} = \mathbb{U}_1 \oplus \mathbb{U}_2 \oplus \cdots \oplus \mathbb{U}_k$ if and only if following two conditions holds;
   (a) $\mathbb{V} = \mathbb{U}_1 + \mathbb{U}_2 + \cdots + \mathbb{U}_k$ and
   (b) $\mathbb{U}_i \cap \mathbb{W}_i = \{0\}$ for each i.
2. Prove that if $\mathbb{W}_1$ and $\mathbb{W}_2$ are finite dimensional subspaces of vector space $\mathbb{V}$, then the subspace $\mathbb{W}_1 + \mathbb{W}_2$ is finite dimensional, and

$$dim(\mathbb{W}_1 + \mathbb{W}_2) = dim(\mathbb{W}_1) + dim(\mathbb{W}_2) - dim(\mathbb{W}_1 \cap \mathbb{W}_2).$$

   Hint: Start with a basis $(u_1, u_2, ..., u_k)$ for $\mathbb{W}_1 \cap \mathbb{W}_2$
3. Let $\mathbb{V}$, $\mathbb{W}$ be finite dimensional vector spaces. Prove that they are isomorphic if and only if $\mathbb{V}$ and $\mathbb{W}$ have the same dimension.
4. Suppose $p \in \mathbb{P}(\mathbb{C})$ has degree m. Prove that p has m distinct roots if and only if p and its derivative p' have no roots in common.
5. Define a linear operator $T : \mathbb{P}_2 \to \mathbb{P}_2$ by $T(p(x)) = p''(x) - p'(x) + p(x)$.
   (a) Find the matrix of T relative to the standard basis $\{1, x, x^2\}$
   (b) Find all eigenvalues of T and the corresponding eigenvectors.
   (c) Is T diagonalizable? Justify your answer.
6. Let $\mathbb{V}$ be a finite dimensional vector space. Suppose that $T \in \mathcal{L}(\mathbb{V})$ has $dim(V)$ distinct eigenvalues and that $S \in \mathcal{L}(\mathbb{V})$ has the same eigenvectors as T (not necessarily with the same eigenvalues). Prove that $ST = TS$.