Analysis Comprehensive Exam

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Instructions. No references are permitted during the exam. Solutions must be written legibly and neatly on separate sheet(s) of paper with your name and problem number at the top of each page. Be sure to provide complete and clear reasons for all of your steps (e.g. no statement is "clear" or "obvious" unless it's a definition or given as an assumption, and do not only name a theorem, also show that its hypotheses are satisfied). All problems are equally weighted. You have three (3) hours to submit your solutions.

In the problems below, all references to "measure", "measurable", "integrable", etc. are with respect to Lebesgue measure on $\mathbb{R}^d$. Also, given $A \subseteq \mathbb{R}^d$, the outer measure of A is denoted by $\boldsymbol{m}^*(A)$, and the Lebesgue measure of A is denoted by $\boldsymbol{m}(A)$.

1. Let X denote the set consisting of sequences $\boldsymbol{x} = (x_n)_{n=1}^\infty$ of real numbers. Define the function $d : X \times X \to \mathbb{R}$ by
$$d(\boldsymbol{x}, \boldsymbol{y}) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}$$
for $\boldsymbol{x} = (x_n)_{n\in\mathbb{N}}$ and $\boldsymbol{y} = \{y_n\}_{n=1}^\infty$ in X. Prove that (X, d) is a metric space.

   Hint: You can use (without proof) that for $a, b, c \geq 0$, if $a \leq b + c$, then $\frac{a}{1+a} \leq \frac{b}{1+b} + \frac{c}{1+c}$.

2. Let $B_{\ell^2} = \{\boldsymbol{x} \in \ell^2 : \|\boldsymbol{x}\|_{\ell^2} \leq 1\}$ denote the closed unit ball of ℓ^2. Prove or disprove that B_{ℓ^2} is compact.

3. Let (X, d) be a metric space and $\{p_n\}_{n=1}^\infty$ be a sequence in X. Define the sets
$$E_N = \{p_n : n \geq N\} \quad \text{for } N \in \mathbb{N}.$$
Prove that $\{p_n\}_{n=1}^\infty$ is Cauchy if and only if $\operatorname{diam}(E_N) \to 0$ as $N \to \infty$.

   Note: As usual, $\operatorname{diam}(S) = \sup\{d(x, y) : x, y \in S\}$ when $S \subseteq X$.

4. Evaluate the limit $\displaystyle\lim_{n\to\infty} \int_0^n \frac{x^{1/n}}{1 + x^2}\, dx$ and prove that your answer is correct, including proving that the limit exists. You can assume basic calculus results.

   Hint: You can use (without proof) the fact that the auxiliary function $\psi(t) = t^{1/t}$ has an absolute maximum on $[1, \infty)$.

5. Let $\mathcal{C}$ denote the Cantor ternary set.
   (a) Describe the main steps in the construction of $\mathcal{C}$.
   (b) Show $\mathcal{C}$ is closed.
   (c) Find the exact value of $\boldsymbol{m}(\mathcal{C})$.

6. Let $E \subseteq \mathbb{R}^d$ be such that $\boldsymbol{m}^*(E) < \infty$. Define $-E = \{-p : p \in E\}$.
   (a) Show that $\boldsymbol{m}^*(-E) = \boldsymbol{m}^*(E)$.
   (b) Use part 6a to prove that if E is Lebesgue measurable, then $-E$ is also Lebesgue measurable.