# ALGEBRA COMPREHENSIVE EXAM - AUGUST 2013

You have three hours to complete this examination. No references or calculators (not needed) are permitted. Please attempt to **answer all questions**, which are equally weighted. You should present the work you wish to be graded on the scratch sheets provided (one sheet per problem). Partial credit will be given. Please try to present your work as legibly as possible. You may use any lemmas or theorems that you can correctly state (other than one you are trying to prove, of course) to establish your arguments. The problems are not presented in any particular order of difficulty, so you should scan the entire list and begin with the ones that seem the easiest to you. Good luck.

1) Find all distinct abelian groups of order $900$ by determining invariant factors, then check the completeness of your list with elementary divisors.

2) Prove Lagrange's Theorem.

3) Show that the Frattini subgroup of $G$ is a characteristic subgroup of $G$. The Frattini subgroup is defined as $\Phi(G) = \bigcap\{H \leq G : H \text{ maximal}\}$. A subgroup $H \leq G$ is maximal in $G$ if whenever $H \leq K \leq G$, then $K = H$ or $K = G$. Recall that a characteristic subgroup is sent back to itself by any automorphism of the containing group.

4) Prove or disprove: $5x^4 + 7x^3 + 11x^2 + 6x + 1$ is irreducible over $\mathbb{Q}$.

5) State which of the following groups are isomorphic and identify the isomorphism. For groups that are not isomorphic, show why no isomorphism exists (you should have six cases total):

$S_3$ (the permutation group on three symbols)
$\mathbb{Z}_6$ (integers modulo 6 under addition)
$GL(2,\mathbb{Z}_2)$ (2 x 2 nonsingular matrices with binary entries under matrix multiplication)
$\mathbb{Z}_2 \times \mathbb{Z}_3$ under (coordinatewise modular) addition.

6) Suppose $R$ is a commutative ring with identity and $I$ is an ideal of $R$. The radical of $I$ is defined to be $\sqrt{I} := \{r \in R \mid r^n \in I$ for some integer $n > 0\}$. Show that $\sqrt{I}$ is an ideal of $R$ containing $I$, and if $P$ is a prime ideal of $R$, then $\sqrt{P} = P$.