Comprehensive Exam – Differential Equations Fall 2013

- Choose ***<u>any six of the following eight</u>*** problems.
- Provide the answer/solution/proof in the method and the format the question requires.
- You may not work out more than one problem per page (one sheet has two pages).
- Write down the answers legibly. Graph the figures as per the instructions.
- Unrecognizable steps/works will not be considered for grading.
- Simplify to the best possible answer. Showing the work is necessary and important. No work means no points.
- Write the question number clearly so that it is still visible even after stapling the exam.
- Nothing is "clear", "obvious" or "trivial" unless it's a definition or given as an assumption.

1. a. State the Picard's existence and uniqueness theorem for the IVP

$$y'(t) = f\big(t, y(t)\big); y(t_0) = x_0.$$

   b. Is the Lipschitz condition in Picard's theorem necessary for the uniqueness of the solution? Justify your answer.

2. Find the general solution of $\boldsymbol{X}'(t) = \boldsymbol{A} \cdot \boldsymbol{X}(t) + \boldsymbol{F}(t)$, where

$$\boldsymbol{A} = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \ and\ \boldsymbol{F}(t) = \begin{pmatrix} e^{4t} \\ 3e^{4t} \end{pmatrix}.$$

3. Consider the competing species system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} x(1 - x - y) \\ y\left(\frac{1}{2} - \frac{y}{4} - \frac{3x}{4}\right) \end{pmatrix}.$$

   a. Find the equilibrium point(s).
   b. Linearize the system for each equilibrium points.
   c. Classify the equilibrium points in terms of their type and stability.
   d. Sketch the phase plane.

4. Using the "geometric" method sketch the phase plane of $x'' + e^x - e = 0$ (Do not use linearization).

5. Consider the system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} y + x - x(x^2 + y^2) \\ y - x - y(x^2 + y^2) \end{pmatrix}.$$

a. Find the equilibrium points.

b. Using the polar coordinates, graph its Phase Plane.

c. Determine, whether this has a limit cycle or not, if it does then discuss the (orbital) stability.

6. Prove or Disprove that $x'' + (\cosh x)x' + e^x = 0$ has a non-constant periodic solutions.

7. Find the series solution of $(x - 5)y' + 2y = 0$ with $y(0) = 1$.

8. Find all the positive eigenvalue(s) and eigenfunction(s) of the (Sturm-Liouville) BVP:

$$y'' + \lambda y = 0;\ 0 < x < 1.$$
$$y'(0) = 0, y(1) = 0.$$

Can $\lambda = 0$ be an eigenvalue, if so find the corresponding eigenfunction, otherwise justify why not?
