Linear Algebra Comprehensive Exam

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Instructions. No references are permitted during the exam. Solutions must be written legibly and neatly on separate sheet(s) of paper with your name and problem number at the top of each page. Be sure to provide complete and clear reasons for all of your steps (e.g. no statement is "clear" or "obvious" unless it's a definition or given as an assumption, and do not only name a theorem, also show that its hypotheses are satisfied). If you are asked to prove a known theorem, do not merely quote that theorem as your proof; instead, produce an independent proof. All problems are equally weighted. You have three (3) hours to submit your solutions.

In the problems below, $\mathbb{F}$ denotes either $\mathbb{C}$ or $\mathbb{R}$, where $\mathbb{R}$ and $\mathbb{C}$ are the fields of real and complex numbers, respectively; $\mathbb{V} \oplus \mathbb{W}$ is the direct sum of the vector spaces $\mathbb{V}$ and $\mathbb{W}$; $\mathcal{L}(\mathbb{V})$ is the set of linear operators $T : \mathbb{V} \to \mathbb{V}$; and $\emptyset$ denotes the null set.

1. Let $\mathbb{V}$ be a vector space. Prove or give a counterexample to the following statement: if $\mathbb{U}_1, \mathbb{U}_2, \mathbb{W}$ are subspaces of $\mathbb{V}$ such that

$$\mathbb{V} = \mathbb{U}_1 \oplus \mathbb{W} \quad \text{and} \quad \mathbb{V} = \mathbb{U}_2 \oplus \mathbb{W}$$

   then $\mathbb{U}_1 = \mathbb{U}_2$.

2. Let $\mathcal{P}_1$ denote the the standard vector space of polynomials $f(t)$ with coefficients in the field of $\mathbb{F}$ and having degree at most 1. Let $\mathcal{S} = \{1, t\}$ be the standard ordered basis of $\mathcal{P}_1$.

   Define $T \in \mathcal{L}(\mathcal{P}_1)$ by

$$T : p(t) = p_0 + p_1 t \longmapsto q(t) = (p_0 - p_1) + (2p_0 + 4p_1)t.$$

   (a) Construct the matrix $[T]_{\mathcal{S}}$ that represents T with respect to the basis $\mathcal{S}$.

   (b) Is there an ordered basis $\mathcal{B}$ for $\mathcal{P}_1$ such that $[T]_{\mathcal{S}}$ is diagonal? If so, give such a basis and the corresponding matrix representation. If not, explain why not.

3. Let $z, \lambda \in \mathbb{F}$. Use mathematical <u>induction</u> to prove that

$$z^j - \lambda^j = (z - \lambda)(z^{j-1} + z^{j-2}\lambda + \cdots + z\lambda^{j-2} + \lambda^{j-1})$$

   for all integers $j \geq 2$.

4. Let $\mathbb{V}$ be a vector space and let $\mathbb{W}_1$ and $\mathbb{W}_2$ be subspaces of $\mathbb{V}$ such that $\mathbb{V} = \mathbb{W}_1 \oplus \mathbb{W}_2$. If β_1 and β_2 are bases for $\mathbb{W}_1$ and $\mathbb{W}_2$, respectively, show that $\beta_1 \cap \beta_2 = \emptyset$ and $\beta_1 \cup \beta_2$ is a basis for $\mathbb{V}$.

5. Let $\mathbb{V}$ and $\mathbb{W}$ be a vector spaces. Prove $\mathbb{V}$ and $\mathbb{W}$ are isomorphic if and only if there is a bijective linear map $T : \mathbb{V} \to \mathbb{W}$.

6. Suppose that $\mathbb{V}$ is a complex inner product space. Prove that

$$\langle u, v \rangle = \frac{\|u + v\|^2 - \|u - v\|^2 + \|u + iv\|^2 i - \|u - iv\|^2 i}{4}$$

   for all $u, v \in \mathbb{V}$, where $\langle u, v \rangle$ denotes the inner product of vector u and v.