Analysis Comprehensive Exam

Department of Chemistry and Mathematics
Florida Gulf Coast University
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Instructions. No references are permitted during the exam. Solutions must be written legibly and neatly on separate sheet(s) of paper with your name and problem number at the top of each page. Be sure to provide complete and clear reasons for all of your steps (e.g. no statement is "clear" or "obvious" unless it's a definition or given as an assumption, and do not only name a theorem, also show that its hypotheses are satisfied). All problems are equally weighted. You have three (3) hours to submit your solutions.

In the problems below, all references to "measure", "measurable", "integrable", etc. are with respect to Lebesgue measure on $\mathbb{R}^d$. Also, given $A \subseteq \mathbb{R}^d$, the exterior measure of A is denoted by $\boldsymbol{m}^*(A)$, and the Lebesgue measure of A is denoted by $\boldsymbol{m}(A)$. The space of continuous functions $C(K)$ on a compact set K is equipped with the norm $\|f\| \stackrel{\text{def}}{=} \sup\{|f(t)| : t \in K\}$, $f \in C(K)$.

1. Let $0 < \alpha < 1$ and $\mathcal{C}_\alpha$ be the subset of $[0,1]$ constructed in an analogous manner as the Cantor set $\mathcal{C}$ except that each of the intervals removed at the nth stage has length $\alpha \cdot 3^{-n}$. Let O_n denote the union of the 2^{n-1} open (disjoint) intervals removed at the nth stage.

   (a) Prove that $\mathcal{C}_\alpha$ is closed, and $\boldsymbol{m}(\mathcal{C}_\alpha) = 1 - \alpha$.

   (b) Define the function $f : [0,1] \to \mathbb{R}$ by setting $f(x) = 1$ when $x \in \mathcal{C}_\alpha$ and $f(x) = 1/2^{n-1}$ when $x \in O_n$ for some $n \in \mathbb{N}$. Show that f is measurable and evaluate (with proof) $\displaystyle\int_{[0,1]} f \, d\boldsymbol{m}$.

2. For each $n \in \mathbb{N}$, let $g_n(x) = \dfrac{nx}{1+nx^2}$ for $x \in [0,1]$. Is the sequence of functions $(g_n)_{n\in\mathbb{N}}$ a Cauchy sequence in $C([0,1])$? Prove that your assertion is correct.

3. Define the functional $\lambda : C[-1,1] \to \mathbb{R}$ by

$$\lambda(f) = \int_{-1}^{0} f(t)\,dt - \int_{0}^{1} f(t)\,dt \quad \text{for } f \in C[-1,1].$$

   Compute (with proof) the operator norm $\|\lambda\|$ of λ.

4. Evaluate the limit

$$\lim_{n\to\infty} \int_0^n \frac{e^{\sin(x^2/n)}}{1+x}\,dx$$

   and prove that your answer is correct, including proving that the limit exists. You can assume basic calculus results.

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5. For $E \subseteq \mathbb{R}^d$ and $x \in \mathbb{R}^d$, let $d(x, E) \stackrel{\text{def}}{=} \inf\{\|x - y\| : y \in E\}$.

   (a) Let $f(x) = d(x, E)$ for $x \in \mathbb{R}^d$. Prove that f is continuous on $\mathbb{R}^d$ and

$$|f(x) - f(y)| \leq \|x - y\| \quad \text{for all } x, y \in \mathbb{R}^d.$$

   (b) Let $n \in \mathbb{N}$ and $U \subseteq \mathbb{R}^d$ be an open set. Prove that the set $C_n \stackrel{\text{def}}{=} \{x \in \mathbb{R}^d : d(x, \mathbb{R}^d \setminus U) \geq 1/n\}$ is a closed subset of $\mathbb{R}^d$ and $C_n \subseteq U$.

   (c) Let $U \subseteq \mathbb{R}^d$ be an open set. Use part 5b to prove that there is a sequence $(K_n)_{n\in\mathbb{N}}$ consisting of compact subsets of $\mathbb{R}^d$ such that

$$K_n \subseteq K_{n+1} \text{ for all } n \in \mathbb{N} \quad \text{and} \quad U = \bigcup_{n\in\mathbb{N}} K_n.$$

   Note: The set U need *not* be bounded.

6. (a) State Tonelli's Theorem for iterated (double) integrals precisely.

   (b) Let E_1 and E_2 be subsets of $\mathbb{R}$. Suppose $E_1 \times E_2$ is a measurable set and $\boldsymbol{m}^*(E_2) > 0$. Prove that E_1 is measurable.

   (c) Give an example (with proof) of two measurable sets A and B such that $A + B$ is not measurable.

   Suggestion: Consider sets of the form $A = \{0\} \times [0, 1]$ and $B = N \times \{0\}$ with an appropriate choice of set N and recall that $A + B \stackrel{\text{def}}{=} \{a + b : a \in A, b \in B\}$.