Comprehensive Exam – Differential Equations Spring 2014

- Choose ***<u>any six from the eight</u>*** the following problems.
- Provide the answer/solutions/proof in the method and the format the question requires.
- You may not work out more than one problem per page (one sheet has two pages).
- Write down the answers legibly. Graph the figures as per the instructions. Calculators are not allowed.
- Unrecognizable steps/works will not be considered for grading.
- Simplify to the best possible. Showing the work is necessary and important. No work means no points.
- Write the question number clearly so that it is still visible even after stapling the exams.
- Nothing is "clear", "obvious" or "trivial" unless it's a definition or given as an assumption.
- If you are using a theorem (or theorems) to justify your arguments, state the theorem(s) in its entirety.

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1. Prove that the system

$$X'(t) = A(t) \cdot X(t)$$

has at most n linearly independent solutions, where $A(t)$ is an $n \times n$ matrix of continuous functions.

2. Find the general solution of $\boldsymbol{X}'(t) = \boldsymbol{A}\boldsymbol{X}(t)$**,** where

$$A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$$

3. Suppose that two species of lizards are in competition, the xanthic lizard (x), and the yellow-bellied lizard (y). A model of the population dynamics involving the interaction between the two species is given by

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} x(0.4 - 0.2x - 0.8y) \\ y(1 - y - x) \end{pmatrix}.$$

   a. Find the critical point(s).
   b. Linearize the system for each critical point.
   c. Classify the critical point(s) in terms of type and stability.
   d. Assume that initially both populations are greater than zero. Indicate whether over time the two species will co-exist, or whether one of the species will go extinct. Justify your answer.

4. Using the "geometric" method sketch the phase plane of $x'' + k^2 \sin x = 0$ (Do not use linearization).

5. Consider the system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} y + x(x^2+y^2)\cos\left(\dfrac{\pi}{\sqrt{x^2+y^2}}\right) \\ -x + y(x^2+y^2)\cos\left(\dfrac{\pi}{\sqrt{x^2+y^2}}\right) \end{pmatrix}.$$

   a. Find the equilibrium points.
   b. Using the polar coordinates, graph its Phase Plane.
   c. Determine, whether this has a limit cycle or not, if it does then discuss the (orbital) stability.

6. Prove or Disprove that $x'' + (x^2 + (x')^2 - 4)x' + x = 0$ has a non-constant periodic solutions.

7. Given the differential equation. $(x-1)y' - y = 0$
   a. Find the series solution for the above equation centered about $x_0 = 3$.
   b. Find the *minimum* radius of convergence for the series solution.

8. Find all the positive eigenvalue(s) greater than three, and the corresponding eigenfunction(s) for the BVP:

$$y'' + (\lambda - 3)y = 0;$$
$$y(0) = 0 = y'(1)$$

Can $\lambda = 3$ be an eigenvalue? If so, find the corresponding eigenfunction; otherwise justify why not.