Linear Algebra Comprehensive Exam

Department of Mathematics
Florida Gulf Coast University
Saturday, January 11, 2014

Instructions. No references are permitted during the exam. Solutions must be written legibly and neatly on separate sheet(s) of paper with your name and problem number at the top of each page. Be sure to provide complete and clear reasons for all of your steps (e.g. no statement is "clear" or "obvious" unless it's a definition or given as an assumption, and do not only name a theorem, also show that its hypotheses are satisfied). If you are asked to prove a known theorem, do not merely quote that theorem as your proof; instead, produce an independent proof. All problems are equally weighted. You have three (3) hours to submit your solutions.

In the problems below, $\mathbb{F}$ denotes either $\mathbb{C}$ or $\mathbb{R}$, where $\mathbb{R}$ and $\mathbb{C}$ are the fields of real and complex numbers, respectively; $\mathbb{V} \oplus \mathbb{W}$ is the direct sum of the vector spaces $\mathbb{V}$ and $\mathbb{W}$; $\mathcal{L}(\mathbb{V})$ is the set of linear operators $T : \mathbb{V} \to \mathbb{V}$; and $\emptyset$ denotes the null set.

1. (a) Prove that if $A \in \mathbb{R}^{n\times n}$ has n distinct eigenvalues λ_j and $|\lambda_j| < 1$, $\forall i$, then $\forall v \in \mathbb{R}^n$:
$$A^N v \to 0 \text{ as } N \to \infty.$$
   (b) Prove that if $A \in \mathbb{R}^{n\times n}$ has n distinct eigenvalues λ_j and $|\lambda_j| > 1$, $\forall i$, then $\forall v \in \mathbb{R}^n$:
$$A^N v \to \infty \text{ as } N \to \infty.$$

2. Define the linear transformation $T \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$ by
$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ -2x_1 + 2x_2 \end{pmatrix}$$
   (a) Is T surjective or not? Show or explain why.
   (b) Is T injective or not? Show or explain why.
   (c) Find $dim(null(T))$.
   (d) Find the matrix of transformation T, that encodes T with respect to the standard (canonical) basis for the domain $\mathbb{R}^2$ and the basis $\{(1,1)^T, (1,-1)^T\}$ for the target space $\mathbb{R}^2$.

3. (a) Show that if M is invertible and similar to K, then K is also invertible, and K^{-1} is similar to M^{-1}.
   (b) Take $A, B \in \mathcal{L}(\mathbb{V}, \mathbb{V})$. Show that if either A or B is invertible, then AB and BA are similar.

4. Let $\mathbb{U}_1, \mathbb{U}_2, \ldots \mathbb{U}_n$ be subspaces of $\mathbb{V}$. Then prove $\mathbb{U}_1 + \mathbb{U}_2 + \cdots + \mathbb{U}_n$ is the smallest subspace of $\mathbb{V}$ that contains all of $\mathbb{U}_1, \mathbb{U}_2, \ldots \mathbb{U}_n$

5. Prove that if $\mathbb{W}_1$ and $\mathbb{W}_2$ are finite dimensional subspaces of vector space $\mathbb{V}$, then the subspace $\mathbb{W}_1 + \mathbb{W}_2$ is finite dimensional, and
$$dim(\mathbb{W}_1 + \mathbb{W}_2) = dim(\mathbb{W}_1) + dim(\mathbb{W}_2) - dim(\mathbb{W}_1 \cap \mathbb{W}_2).$$
   Hint: You may start with a basis $(u_1, u_2, ..., u_k)$ for $\mathbb{W}_1 \cap \mathbb{W}_2$

6. Prove the following theorem. Let $\mathbb{V}$ be a vector space with subspaces $\mathbb{U}$ and $\mathbb{W}$. Then $\mathbb{V} = \mathbb{U} \oplus \mathbb{W}$ **iff** $\mathbb{V} = \mathbb{U} + \mathbb{W}$ and $\mathbb{U} \cap \mathbb{W} = \{0\}$.