Modern Algebra Comprehensive Exam
August 2014

You have three hours to complete this examination. No references or calculators are allowed. Please select six out of the eight questions to be graded. All questions are equally weighted. You should present the work you wish to be graded on the scratch sheets provided (one sheet per problem). Partial credit will be given. Please try to present your work as legibly as possible. You may use any lemmas or theorems that you can correctly state, but you must show that all conditions of those theorems are satisfied to establish your arguments. The problems are not presented in any particular order of difficulty, so you should scan the entire list and begin with the ones that seem the easiest to you. Good luck!

Problem 1. Prove the First Isomorphism Theorem of Groups: Let G and H be groups, and let $\phi : G \to H$ be a group homomorphism. Then

(a) The kernel of ϕ is a normal subgroup G,
(b) The image of ϕ is a subgroup of H, and
(c) The image of ϕ is isomorphic to the quotient group $H/ker(\phi)$.

Problem 2. Prove that no group of order 380 is simple.

Problem 3.

(a) Prove or disprove: characteristic subgroups are normal
(b) Prove or disprove: normal subgroups are characteristic.
(c) Prove that if H is the unique subgroup of a given order in a group G then H is characteristic in G.

Problem 4. Let F be a field and x an indeterminate over F. Let $f(x)$ be a polynomial of degree $n \geq 1$ and for $g(x) \in F[x]$ let $\overline{g(x)} = g(x) + (f(x))$ denote the image of $g(x)$ in the quotient ring $F[x]/(f(x))$.

(a) Prove that $\overline{1}, \overline{x}, \overline{x^2}, \ldots, \overline{x^{n-1}}$ is a basis for $F[x]/(f(x))$.
(b) How many elements does $F[x]/(f(x))$ have if F is a finite field of order q?
(c) Find a field F and a polynomial $f(x) \in F[x]$ such that the field $F[x]/(f(x))$ has exactly 121 elements.

Problem 5.

(a) Let R be an integral domain. Prove that $(a) = (b)$ for some elements a and b in R if and only if $a = ub$ for some unit u of R.
(b) Let R be the ring of all continuous functions on $[0, 1]$ and let I be the collection of all functions $f(x)$ in R such that $f(1/5) = f(1/3) = 0$. Prove that I is an ideal and that I is not prime.

Problem 6.

(a) Determine the Galois Group of $p(x) = (x^2 + 2)(x^2 - 3)$.
(b) Determine *all* of the subfields of the splitting field of this polynomial and draw the corresponding lattice of subfields and their stabilizing subgroups.