Analysis Comprehensive Exam

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Instructions. No references are permitted during the exam. Solutions must be written legibly and neatly on separate sheet(s) of paper with your name and problem number at the top of each page. Be sure to provide complete and clear reasons for all of your steps (e.g. no statement is "clear" or "obvious" unless it is a definition or given as an assumption, and do not only name a theorem, also show that its hypotheses are satisfied). All problems are equally weighted. You have three (3) hours to submit your solutions.

In the problems below, all references to "measure", "measurable", "integrable", etc. are with respect to Lebesgue measure on $\mathbb{R}^d$. Also, given $A \subseteq \mathbb{R}^d$, the exterior measure of A is denoted by $\boldsymbol{m}^*(A)$, and the Lebesgue measure of A is denoted by $\boldsymbol{m}(A)$.

1. (a) Let $E \subseteq \mathbb{R}^d$. Prove that if E is a Lebesgue measurable set, then χ_E is a measurable function.

   (b) Give an example (*with proof*) of a function $f : \mathbb{R} \to \mathbb{R}$ which is *not* a measurable function.

2. Let $(E_n)_{n\in\mathbb{N}}$ be a sequence of measurable sets in $\mathbb{R}^d$ such that $\boldsymbol{m}(E_n) < 1/n$ for all $n \in \mathbb{N}$. Consider the function $\varphi : \mathbb{R}^d \to [0, \infty]$ defined by

$$\varphi(x) = \sum_{n=1}^{\infty} \frac{1}{n} \chi_{E_n}(x) \qquad \text{for } x \in \mathbb{R}^d.$$

   (a) Show that φ is a measurable function.

   (b) Is the function φ integrable? Justify your reasoning.

3. (a) Give an example (with proof) of a function $f : [0, 1] \to \mathbb{R}$ which is (Lebesgue) integrable but *not* Riemann integrable.

   (b) Let f be an integrable function on $[0, 1]$. Prove that $\displaystyle\lim_{n\to\infty} \int_{[0,1]} x^n\, f(x)\, d\boldsymbol{m}(x) = 0$.

4. Consider $\mathbb{R}^2$ equipped with the ℓ^∞-norm; that is, $\|x\|_{\ell^\infty} = \max\{|x_1|, |x_2|\}$ for $x = (x_1, x_2) \in \mathbb{R}^2$.

   (a) Show that no inner product on $\mathbb{R}^2$ induces the ℓ^∞-norm; that is, there is no positive-definite symmetric bilinear function $\langle \cdot, \cdot \rangle$ defined on $\mathbb{R}^2$ such that

$$\|x\|_{\ell^\infty} = \sqrt{\langle x, x \rangle} \text{ for all } x \in \mathbb{R}^2.$$

   (b) Consider the set

$$A = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 1\}.$$

   Prove that A is convex and contains infinitely many points which are at minimal distance from the origin.

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5. Let $C[0,1]$ be equipped with its usual norm $\|f\| \stackrel{\text{def}}{=} \sup\{|f(t)| : t \in [0,1]\}$. For each $n \in \mathbb{N}$, let $g_n(x) = \dfrac{nx}{1+nx^3}$ for $x \in [0,1]$. Is the sequence of functions $(g_n)_{n\in\mathbb{N}}$ a Cauchy sequence in $C([0,1])$? Prove that your assertion is correct.

6. Let (X,d) be a metric space. For $r > 0$ and $x \in X$, let $B(x;r) = \{y \in X : d(x,y) < r\}$ and $C(x;r) = \{y \in X : d(x,y) \leq r\}$ denote the open and closed balls centered at x of radius r, respectively. For $A \subseteq X$, let $\operatorname{clos}(A)$ denote the closure of the set A.

   (a) Prove that $\operatorname{clos}(B(x;r)) \subseteq C(x;r)$.

   (b) Suppose now $(X, \|\cdot\|)$ is a *normed* space equipped with its usual metric $d(x,y) = \|x-y\|$ for $x, y \in X$. Prove that $C(x;r) \subseteq \operatorname{clos}(B(x;r))$.