Comprehensive Exam – Differential Equations Fall 2014

- Choose ***any six from the eight*** the following problems.
- Provide the answer/solutions/proof in the method and the format the question requires.
- You may not work out more than one problem per page (one sheet has two pages).
- Write down the answers legibly. Graph the figures as per the instructions.
- Unrecognizable steps/works will not be considered for grading.
- Simplify to the best possible. Showing the work is necessary and important. No work means no points.
- Write the question number clearly so that it is still visible even after stapling the exams.
- Nothing is "clear", "obvious" or "trivial" unless it's a definition or given as an assumption.
- If you are using a theorem (or theorems) to justify your arguments, state the theorem(s) in its entirety.

---

1. A) State Picard's existence & uniqueness theorem.

   B) Consider the Initial Value Problem (IVP):

$$x'(t) = \begin{cases} -t \cdot x^{\frac{1}{2}}\,; x \geq 0 \\ t|x|^{\frac{1}{2}}\,; x < 0 \end{cases}, \quad x(0) = 0.$$

   Does this problem satisfy Picard's theorem? Explain

   Does this problem have a unique solution? Justify your answer.

2. Solve $\boldsymbol{X}'(t) = \boldsymbol{AX}(t)$, where

$$A = \begin{pmatrix} -2 & -1 & \alpha \\ 1 & -4 & 1 \\ 0 & 0 & -3 \end{pmatrix}$$

   When

   $A)$ $\alpha = 1$ and

   B) $\alpha \neq 1$.

3. Consider the competing species system:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} x(1 - x - y) \\ y\left(\frac{1}{2} - \left(\frac{1}{4}\right)y - \left(\frac{3}{4}\right)x\right) \end{pmatrix}.$$

A) Find the critical point(s).

B) Linearize the system for each critical point.

C) Classify the critical point(s) in terms of type and stability.

D) Assume that initially both populations are greater than zero. Indicate whether over time the two species will co-exist, or whether one of the species will go extinct. Justify your answer.

4. Consider the IVP

$$x'' + x = x^2 = 0$$
$$x(0) = \alpha, x'(0) = \beta.$$

A) If $\alpha = 1, \beta = 0$ compute $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} x'(t)$

B) If $\alpha = 0, \beta = -\frac{1}{\sqrt{3}}$ compute $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} x'(t)$

C) If $\alpha = 1, \beta = -2$ compute $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} x'(t)$

D) Are there any periodic solutions? If so, what should α and β satisfy for the solution to be periodic.

5. Show that the system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -y + \dfrac{xf(r)}{r} \\ x + \dfrac{yf(r)}{r} \end{pmatrix}$$

   A) has periodic solutions corresponding to the zeros of $f(r)$. What is the direction of motion on the closed trajectories in the phase plane.
   B) Let $f(r) = r(r-1)^2(r^2 - 4r + 3)$. Determine all periodic solutions and their stability characteristics.

6. Prove or Disprove that $x'' + e^x x' + x = 0$ has a non-constant periodic solutions.

7. Consider the Hermite's equation $y'' - 2xy' + \lambda y = 0$. Find the series solution of the special case when $\lambda = 1$.

8. Find all the positive eigenvalue(s), and the corresponding eigenfunction(s) for the BVP:

$$y'' + \lambda y = 0;$$
$$y'(0) = 0 = y'(1)$$

Can $\lambda = 0$ be an eigenvalue? If so, find the corresponding eigenfunction; otherwise justify why not.