Linear Algebra Comprehensive Exam

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Saturday, August 23, 2014

Instructions. No references are permitted during the exam. Solutions must be written legibly and neatly on separate sheet(s) of paper with your name and problem number at the top of each page. Be sure to provide complete and clear reasons for all of your steps (e.g., no statement is “clear” or ”obvious” unless it is a definition or given as an assumption; and do not only name a theorem, but also show that its hypotheses are satisfied). If you are asked to prove a known theorem, do not merely quote that theorem as your proof; instead, produce an independent proof. Do not interpret a problem in such a way that it becomes trivial. All problems are equally weighted, though each of the same problem need not be. You have three (3) hours to submit your solutions to the following six (6) problems.

Notation. The fields of complex and real numbers are denoted by $\mathbb{C}$ and $\mathbb{R}$, respectively. Throughout, $M_n(\mathbb{F})$ denotes the space of $n \times n$ matrices with entries from the field $\mathbb{F}$. We denote $M_n(\mathbb{C})$ by M_n. The vector space of n-tuples with entries from a field $\mathbb{F}$ is denoted by $\mathbb{F}^n$. Matrices are specified by capital letters, such as A or B, and vectors by lowercase letters, such as x or y. If we wish to specify the entries of a matrix or vector, we will write $A = [a_{ij}]$ or $x = [x_i]$.

1. Suppose V is a finite dimensional vector space over a field, and U and W are subspaces of V. Denote the direct sum of U and W by $U \oplus W$. Prove that $V = U \oplus W$ if and only if $U \cap W = \{0\}$ and $\dim V = \dim U + \dim W$.

2. Let $A \in M_n$ be given, and suppose that $A^k = 0$ for some $k > n$.
   (a) Prove that 0 is the only eigenvalue of A.
   (b) Using the Jordan canonical form, prove that $A^r = 0$ for some $r \leq n$.

3. Recall that the l_1-norm for $A = [a_{ij}] \in M_n$ is defined by
$$||A||_1 = \sum_{i,j=1}^{n} |a_{ij}| = \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|.$$
   Prove that the l_1-norm is a matrix norm on M_n.

4. The following parts are unrelated and equally weighted:
   (a) Let $\mathcal{P}$ denote the vector space of all polynomials with complex coefficients and $S = \{z_1, z_2, \ldots, z_n\} \subseteq \mathbb{C}$. Define $V = \{f \in \mathcal{P} : f(z_i) = 0 \text{ for } 1 \leq i \leq n\}$. Prove that V is a subspace of $\mathcal{P}$.
   (b) Let $A \in M_n$. Prove that if A has all nonzero singular values then A has all nonzero eigenvalues.

5. A matrix $A \in M_n$ is a *square root* of $B \in M_n$ if $A^2 = B$.
   (a) Show that every diagonalizable matrix in M_n has a square root in M_n.
   (b) Is this true in $M_n(\mathbb{R})$? That is, does every diagonalizable matrix in $M_n(\mathbb{R})$ have a square root in $M_n(\mathbb{R})$? Prove or provide a counterexample.

6. Suppose $A \in M_n$ is Hermitian. Prove that A is positive definite if and only if A has all positive eigenvalues. [*Hint*: for the “$\Longleftarrow$” direction, use the Spectral Theorem for Hermitian matrices.]