# Abstract Algebra Comprehensive Exam

January 2015

**Name:** ____________________________

You have three hours to complete this examination. No references or calculators are allowed. Please complete all six problems. All questions are equally weighted. You should present the work you wish to be graded on the scratch sheets provided (one sheet per problem). Partial credit will be given. Please try to present your work as legibly as possible. You may use any lemmas or theorems that you can correctly state, but you must show that all conditions of those theorems are satisfied to establish your arguments. The problems are not presented in any particular order of difficulty, so you should scan the entire list and begin with the ones that seem the easiest to you. Good luck!

**Problem 1.**[20 points] State and prove Lagrange's Theorem on the cardinality of subgroups.

**Problem 2.** Let $G$ be a group with $|G| = 105$.

(a) Prove $G$ has a normal Sylow 5-subgroup or a normal Sylow 7-subgroup.

(b) Go one step further, and show that $G$ has a normal 5-subgroup **and** a normal Sylow 7-subgroup.

**Problem 3.** Find all distinct abelian groups of order 360 by determining invariant factors, then check the completenes of your list with elementary divisors.

**Problem 4.**

(a) Prove that if $H$ is the unique subgroup of a given order in a group $G$ then $H$ is characteristic in $G$.

(b) Prove that if $H$ is a characteristic subgroup of $N$ and $N$ is a normal subgroup of $G$, then $H$ is a normal subgroup of $G$.

**Problem 5.** Show that $f(x) = x^4 + 1$ is irreducible over $\mathbb{Q}$.

**Problem 6.** Let $K$ be a field. Prove or disprove:

(a) Prove that $K[t]$ is a Euclidean domain.

(b) Prove that every Euclidean domain is a principal ideal domain.