Analysis Comprehensive Exam

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Instructions. No references are permitted during the exam. Solutions must be written legibly and neatly on separate sheet(s) of paper with your name and problem number at the top of each page. Be sure to provide complete and clear reasons for all of your steps (e.g. no statement is "clear" or "obvious" unless it is a definition or given as an assumption, and do not only name a theorem, also show that its hypotheses are satisfied). All problems are equally weighted. You have three (3) hours to submit your solutions.

In the problems below, all references to "measure", "measurable", "integrable", etc. are with respect to Lebesgue measure on $\mathbb{R}^d$. Also, given $A \subseteq \mathbb{R}^d$, the exterior measure of A is denoted by $\boldsymbol{m}^*(A)$, and the Lebesgue measure of A is denoted by $\boldsymbol{m}(A)$. As usual, $B(x;r) = \{y : d(x,y) < r\}$ denotes the open ball centered at x of radius r.

1. Let $\mathcal{P}_1$ denote the collection of polynomials of degree at most 1. Recall that for $p, q \in C[0,1]$,

$$\langle p, q \rangle \stackrel{\text{def}}{=} \int_0^1 p(t)\overline{q(t)}\, dt.$$

   (a) Show that there is a unique polynomial p of degree at most one (1) such that $\displaystyle\int_0^1 \left|t^3 - p(t)\right|^2 dt$ is minimal.

   (b) Find the unique polynomial p of degree at most one (1) such that $\displaystyle\int_0^1 |t^3 - p(t)|^2\, dt$ is minimal.

2. Let Λ be a (non-empty) set and $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Suppose that $(x_n)_{n\in\mathbb{N}}$ converges to x_0 in $\mathbb{R}$ and let $K = \{x_0, x_1, \ldots\}$. Prove that if $(O_\alpha)_{\alpha\in\Lambda}$ is an open cover for K, then there are $\alpha_0, \alpha_1, \ldots, \alpha_N \in \Lambda$ such that

$$K \subseteq O_{\alpha_0} \cup O_{\alpha_1} \cup \ldots \cup O_{\alpha_N}.$$

3. Let $C[0,1]$ be equipped with its usual norm $\|f\| \stackrel{\text{def}}{=} \sup\{|f(t)| : t \in [0,1]\}$. For each $n \in \mathbb{N}$, let $g_n(x) = \sin^n(\pi x)$ for $x \in [0,1]$. Is the sequence of functions $(g_n)_{n\in\mathbb{N}}$ a Cauchy sequence in $C[0,1]$? Prove that your assertion is correct.

4. Let $\mathcal{K}$ be the subset of $[0,1]$ constructed in an analogous manner as the Cantor ternary set $\mathcal{C}$ except that each of the open intervals removed at the nth stage has length $2/7^n$. Let $\mathcal{O}_n$ denote the union of the 2^{n-1} open disjoint intervals removed at the nth stage.

   Provide an inductive construction of $\mathcal{K}$, show that $\mathcal{K}$ is Lebesgue measurable, and find the *exact* value of $\boldsymbol{m}(\mathcal{K})$.

Continued on the next page.

5. Evaluate the limit

$$\lim_{n\to\infty} \int_{-1}^{1} \frac{x^{3n}}{1+x^{2n}}\, dx$$

and prove that your answer is correct, including proving that the limit exists. You can assume basic calculus results.

6. Define $\varphi : \mathbb{R} \to \mathbb{R}$ by $\varphi(x) = n^2$ if $n \le x < n + \dfrac{1}{n^3}$ for some $n \in \mathbb{N}$ and $\varphi(x) = 0$ otherwise.

   (a) Show that φ is measurable.

   (b) Does φ belong to $L^1(\boldsymbol{m})$? Prove that your answer is correct.