Comprehensive Exam – Differential Equations Spring 2015

- Choose ***<u>any six from the eight</u>*** questions.
- Cross out the entire page of the problem that you don't want to be graded.
- Provide the answer/solutions/proof in the method and the format the question requires.
- Write down the answers legibly. Graph the figures as per the instructions.
- Unrecognizable steps/works will not be considered for grading.
- Simplify to the best possible. Showing the work is necessary and important. No work means no points.
- Nothing is "clear", "obvious" or "trivial" unless it's a definition or given as an assumption.
- If you are using a theorem (or theorems) to justify your arguments, state the theorem(s) in its entirety.
- Any type of calculator or computing device is not allowed.

For Instructor's Use Only:

Questions	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Total Score
Points									

1. A. [6 points] State the Picard's existence and uniqueness theorem and prove its uniqueness part.

B. [4 points] Consider the following IVP:

$$y'(x) = \frac{\sqrt{x+y}}{x-y}, y(0) = 1 \ for \ x \in \left(-\frac{1}{3}, \frac{1}{3}\right).$$

Does it have a unique solution on the open square $\left(-\frac{1}{3}, \frac{1}{3}\right) \times \left(\frac{2}{3}, \frac{4}{3}\right)$? Justify your answer.

2. Solve the IVP $\boldsymbol{X}' = \boldsymbol{A}(\boldsymbol{t})\boldsymbol{X} + \boldsymbol{g}(\boldsymbol{t})$ where $A(t) = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$ and $g(t) = \begin{pmatrix} 0 \\ \cos t \end{pmatrix}$.

3. Consider the system $\boldsymbol{X}'(t) = \boldsymbol{AX}(t)$, where $A = \begin{pmatrix} 0 & -5 \\ 1 & \alpha \end{pmatrix}$.

   A. [3 points] Determine the eigenvalues of A in terms of α

   B. [4 points] Determine the type and stability of the equilibrium point (0,0) for all $\alpha \in \mathbb{R}$.

   C. [3 points] Draw the phase plane (or phase portrait) for the corresponding α.

4. Prove that every solution of $x'' + x^5 = 0$ is periodic.

5. Consider the system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} y + x(x^2 + y^2)\sin\left(\dfrac{\pi}{\sqrt{x^2 + y^2}}\right) \\ -x + y(x^2 + y^2)\sin\left(\dfrac{\pi}{\sqrt{x^2 + y^2}}\right) \end{pmatrix}.$$

a. Find the equilibrium points.

b. Using the polar coordinates, graph its Phase Plane.

c. Determine, whether this has a limit cycle or not, if it does then discuss the (orbital) stability.

6. Graph the phase plane of $x'' - e^{-x} = 0$ using the "geometric method".

7. Find the series solution of $(x^2 + 1)y'' + xy' - y = 0$ centered about the origin.

8. Find all the positive eigenvalue(s), and the corresponding eigenfunction(s) for the BVP:

$$y'' + \lambda y = 0;$$
$$y(0) = 0 = y'(L)$$

Can $\lambda = 0$ be an eigenvalue? If so, find the corresponding eigenfunction; otherwise justify why not.