Abstract Algebra Comprehensive Exam
August 2015

Name: ____________________________

You have three hours to complete this examination. No references or calculators are allowed. Please complete all six problems. All questions are equally weighted. You should present the work you wish to be graded on the scratch sheets provided (one sheet per problem). Partial credit will be given. Please try to present your work as legibly as possible. You may use any lemmas or theorems that you can correctly state, but you must show that all conditions of those theorems are satisfied to establish your arguments. The problems are not presented in any particular order of difficulty, so you should scan the entire list and begin with the ones that seem the easiest to you. Good luck!

Problem 1. Let G be a group and $H \leq G$ a subgroup.

(a) Prove that if H is the unique subgroup of a given order in a group G then H is characteristic in G.

(b) Prove that if H is a characteristic subgroup of N and N is a normal subgroup of G, then H is a normal subgroup of G.

Problem 2. Classify the groups of order 28 (there are four isomorphism types).

Problem 3. Let G be a group with $|G| = 231$.

(a) Prove G has a normal Sylow 7-subgroup or a normal Sylow 11-subgroup.

(b) Go one step further, and show that G has a normal 7-subgroup **and** a normal Sylow 11-subgroup.

Problem 4. Let F be a field. Prove that the set $R = \left\{a_0 + a_2x^2 + \cdots + a_nx^n \mid a_i \in F\right\}$ of polynomials in $F[x]$ whose coefficient of x is equal to 0 is a subring of $F[x]$ and that R is not a U.F.D.

Problem 5. Let F denote the splitting field of $p(x) = x^4 - 2$ over $\mathbb{Q}$. Determine the generators for F and its degree over $\mathbb{Q}$.

Problem 6. Prove that any algebraically closed field is infinite.