Analysis Comprehensive Exam

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Instructions. No references are permitted during the exam. Solutions must be written legibly and neatly on separate sheet(s) of paper with your name and problem number at the top of each page. Be sure to provide complete and clear reasons for all of your steps (e.g. no statement is "clear" or "obvious" unless it is a definition or given as an assumption, and do not only name a theorem, also show that its hypotheses are satisfied). All problems are equally weighted. You have three (3) hours to submit your solutions.

In the problems below, $\boldsymbol{m}$ denotes Lebesgue measure on $\mathbb{R}$.

1. Let $C[0,1]$ be equipped with its usual supremum norm. For each $n \in \mathbb{N}$, let $g_n(x) = \sin^n(\pi x)$ for $x \in [0,1]$. Is the sequence of functions $(g_n)_{n\in\mathbb{N}}$ a Cauchy sequence in $C[0,1]$? Make a claim and prove that your assertion is correct.

2. Suppose X is a metric space and K is a compact subset of X. Fix $\epsilon > 0$. Prove that there is an $N \in \mathbb{N}$ such that for any $S \subseteq K$ with N points there are two points $x, y \in S$ with $d(x,y) < \epsilon$.

   Suggestion: Consider open balls of radius $\epsilon/2$.

3. Let X and Y be normed spaces. Let B_X and B_Y denote the *closed* unit balls in X and Y, respectively. Suppose $T : X \to Y$ is a linear operator and that there is an $r > 0$ such that

   $$r \cdot B_Y \subseteq T(B_X).$$

   Prove that for any $y \in Y$, there is an $x \in X$ such that

   $$y = Tx \quad \text{and} \quad \|x\| \leq M\|y\|,$$

   where M is a constant independent of y.

4. Evaluate the limit

   $$\lim_{n\to\infty} \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-x}\, d\boldsymbol{m}(x)$$

   and prove that your answer is correct, including proving that the limit exists.

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5. Let $(X, \mathcal{A}, \mu)$ be a measure space and $(E_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$. Suppose $E_{n+1} \subseteq E_n$ holds for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} \mu(E_n) = 0$.

   (a) Let $E = \bigcap\limits_{n\in\mathbb{N}} E_n$. Prove that $\lim\limits_{n\to\infty} \chi_{E_n}(t) = \chi_E(t)$.

   (b) Suppose $g \in L^1(\mu)$. Prove that $\lim\limits_{n\to\infty} \int_{E_n} f\, d\mu = 0$.

6. Let $\mathcal{K}$ be the subset of $[0,1]$ constructed in an analogous manner as the Cantor ternary set $\mathcal{C}$ except that each of the open intervals removed at the nth stage has length $\dfrac{2}{5^n}$. Do all of the following:

   (a) Provide an inductive construction of $\mathcal{K}$,

   (b) show that $\mathcal{K}$ is Lebesgue measurable, and

   (c) find the *exact* value of $\boldsymbol{m}(\mathcal{K})$.