Comprehensive Exam - Differential Equations, Fall 2015

Date: ______________ Name:________________________

Directions:

- Choose **any six (6) from eight (8)** of the following problems. Each problem worth 10 points.
- Provide the answer/solutions/proof in the method and the format the question requires.
- You may not work out more than one problem per page (one sheet has two pages).
- Write down the answers legibly. Graph the figures as per the instructions.
- Unrecognizable steps/works will not be considered for grading.
- Simplify to the best possible. Showing the work is necessary and important. No work means no points.
- Write the question number clearly so that it is still visible even after stapling the exams.
- Nothing is "clear", "obvious" or "trivial" unless it's a definition or given as an assumption.
- If you are using a theorem (or theorems) to justify your arguments, state the theorem(s) in its entirety.
- Here "IVP" means Initial Value Problem, and "BVP" means Boundary Value Problem.
- No calculators allowed.

For Grading Purpose Only

Solution # (Question No:)	Score
Solution 1 (Qn.No:)	
Solution 2 (Qn.No:)	
Solution 3 (Qn.No:)	
Solution 4 (Qn.No:)	
Solution 5 (Qn.No:)	
Solution 6 (Qn.No:)	
Overall:	

Q1: (a) (3 points) State Picard's existence and uniqueness theorem.

(b) (7 points) Prove or disprove that the IVP $y'(t) = 5y^{\frac{1}{5}}(t), y(0) = 0$ has a unique solution.

Q2: Convert the differential equation $x''(t) + 4x(t) = t$ to a system of differential equations and find the general solution of the system using eigenvalue-eigenvector method.

Q3: Show that $\Phi(t) = \begin{pmatrix} t^2 & t \\ 2t & 1 \end{pmatrix}$ is a fundamental matrix for the system $\mathbf{X}'(t) = \mathbf{A}(t) \cdot \mathbf{X}(t)$,
where $\mathbf{A}(\mathbf{t}) = \begin{pmatrix} 0 & 1 \\ -\frac{2}{t^2} & \frac{2}{t} \end{pmatrix}$, for $t \in (a, b)$ with $ab > 0$.

Q4: Consider the system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} y + x - x(x^2 + y^2) \\ y - x - y(x^2 + y^2) \end{pmatrix}$$

(a) Find the equilibrium points

(b) Using polar coordinates, graph its Phase Portrait

(c) Determine, whether this system has a limit cycle or not, if it does, then discuss the (orbital) stability.

Q5: Consider $y'' + |y| = 0,\ y(0) = 0, y(b) = B$, with $b > 0$. Determine the number of solutions for all possible positive values of b and B.

Q6: Find all positive eigenvalues and eigenfunctions for the (Sturm-Liouville) BVP:

$$y'' + \lambda y = 0,\ 0 < x < 1,$$
$$y(0) = 0,\ y(1) = 0.$$

Q7: Prove or disprove that $x'' + (\cosh x)x' + x = 0$ has a non-constant periodic solution.

Q8: Answer All: Fill in the Blanks and True or False (No justification necessary)

(a) True or False: The IVP $y' = (y - 6)(y - 11), y(0) = 7$ has a unique solution $y(t)$ such that $\lim_{t\to-\infty} y(t) = 11$.
Circle your answer: T / F

(b) True or False: The IVP

$$y'(t) = e^{t^3} \sin(y) + \cos t \sin(e^t + y - 1), y(0) = 1$$

has a unique solution $y(t)$ that exists for all $t \in \mathbb{R}$, and is such that $y(t) \neq 0$ for all $t \in \mathbb{R}$.
Circle your answer: T / F

(c) Fill in the blanks: If $\underline{A} = \begin{pmatrix} 5 & 0 \\ 0 & 7 \end{pmatrix}$, then $e^{\underline{A}t} =$ ____________

(d) Fill in the blanks: Consider the system $\mathbf{X}'(t) = \mathbf{A} \cdot \mathbf{X}(t)$, where $\mathbf{A} = \begin{pmatrix} 0 & 5 \\ -1 & -2 \end{pmatrix}$.

i. The type of the critical point $(0, 0)$ is ____________

ii. The stability of the critical point $(0, 0)$ is ____________

(e) Fill in the blanks: Consider $x'' + e^x = 0, x(0) = 0, x'(0) = 4$. Evaluate:

i. $\lim_{t\to\infty} x(t) =$ ________

ii. $\lim_{t\to\infty} x'(t) =$ ________.

(Hint: Use Phase Plane)