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Linear Algebra

August 2015

Name : ______________________________

Comprehensive Exam

Exam Instructions

- No references are permitted during the exam.
- Solutions must be written legibly and neatly in the space provided for each problem.
- Be sure to provide complete and clear reasons for all of your steps (e.g., no statement is “clear” or ”obvious” unless it is a definition or given as an assumption; and do not only name a theorem, but also show that its hypotheses are satisfied).
- If you are asked to prove a known theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Do not interpret a problem in such a way that it becomes trivial.
- All problems are equally weighted.
- You have three (3) hours to submit your solutions to the following six (6) problems.
- Students are encouraged to use a scientific calculator such as TI-83/TI-84/TI-89.

Problem-1

Let $a \in \mathbb{R}$, $x = \begin{bmatrix} a \\ 1 \\ 3 \\ 0 \end{bmatrix}$, $u = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$, and $v = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$. Let $\mathbb{P}_u(x)$ and $\mathbb{P}_v(x)$ be the projections of x onto u and v respectively. Find the values of a such that $\mathbb{P}_u(x)$ is orthogonal to $\mathbb{P}_v(x)$.

Problem-2

Find an orthonormal basis in $\mathbb{R}^3$ using the set of basis vectors $\left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Problem-3

Let $a \in \mathbb{R}, x = \begin{bmatrix} a \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $u^{\perp} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$. Compute the Householder reflection of x about the hyperplane with the orthogonal vector $u^{\perp}$.

Problem-4

Suppose A is an $n \times n$ square matrix such that $A^2 = 0$. Prove that $\sigma(A) = \{0\}$.

Problem-5

Suppose A is an $m \times m$ normal matrix. Prove that the following two statements are equivalent:

(1) A has a singular value decomposition $A = U\Sigma V^T$, where U=V.

(2) A is symmetric and positive semi-definite.

Problem-6

Compute $\ln(A)$ if $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$