Abstract Algebra Comprehensive Exam

January 2016

Name: ____________________________

You have three hours to complete this examination. No references or calculators are allowed. Please complete all six problems. All questions are equally weighted. You should present the work you wish to be graded on the scratch sheets provided (one sheet per problem). Partial credit will be given. Please try to present your work as legibly as possible. You may use any lemmas or theorems that you can correctly state, but you must show that all conditions of those theorems are satisfied to establish your arguments. The problems are not presented in any particular order of difficulty, so you should scan the entire list and begin with the ones that seem the easiest to you. Good luck!

Problem 1. Prove the First Isomorphism Theorem of Groups: Let G and H be groups, and let $\phi : G \to H$ be a group homomorphism. Then

(a) The kernel of ϕ is a normal subgroup G,
(b) The image of ϕ is a subgroup of H, and
(c) The image of ϕ is isomorphic to the quotient group $G/\ker(\phi)$.

Problem 2. Let p, q, and r be primes with $p < q < r$. Prove that any G with $|G| = pqr$ is not simple.

Problem 3. Prove that if R is a P.I.D. and D is a multiplicatively closed subset of R, then $D^{-1}R$ is also a P.I.D.

Problem 4. Find all distinct abelian groups of order 360 by determining invariant factors, then check the completenes of your list with elementary divisors.

Problem 5.

(a) Determine the Galois Group of $f(x) = (x^2 - 3)(x^2 - 5)$.

(b) Determine *all* of the subfields of the splitting field of this polynomial and draw the corresponding lattices of subfields and their stabilizing subgroups.

Problem 6. Let K be a field.

(a) Prove that $K[t]$ is a Euclidean domain.

(b) Prove that every Euclidean domain is a principal ideal domain.