Analysis Comprehensive Exam

Department of Mathematics
Florida Gulf Coast University
Friday, January 8, 2016

Instructions. Please **write your name at the top of every page** and look to the board for possible corrections.

No references are permitted during the exam. Solutions must be written legibly and neatly on separate sheet(s) of paper with your name and problem number at the top of each page. Be sure to provide complete and clear reasons for all of your steps (e.g. no statement is "clear" or "obvious" unless it is a definition or given as an assumption, and do not only name a theorem, also show that its hypotheses are satisfied). All problems are equally weighted. You have three (3) hours to submit your solutions.

Problem	Score (out of 10 points)
1	
2	
3	
4	
5	
6	

Name: ____________________ **Final Score:** ____________

Note: Throughout, $\boldsymbol{m}$ and $\mathcal{M}$ denote Lebesgue measure on $\mathbb{R}$ and the collection of all Lebesgue measurable subsets of $\mathbb{R}$, respectively.

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1. Suppose X and Y are normed spaces and $T : X \to Y$ is a surjective bounded linear operator. If X is complete and there is a constant $K > 0$ such that

$$\|x\| \leq K\|Tx\| \quad \text{for all } x \in X,$$

prove that Y is complete.

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2. Let X be a set, $K \subseteq X$, and d be the *discrete* metric on X, i.e.

$$d(x,y) = \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}.$$

Find necessary and sufficient conditions for K to be compact. Then prove that you are correct.

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3. Let $C[0,1]$ be equipped with its usual supremum norm. For each $n \in \mathbb{N}$, let

$$f_n(x) = \begin{cases} 2^n x & \text{if } 0 \leq x \leq \frac{1}{2^n} \\ 1 & \text{if } \frac{1}{2^n} < x \leq 1 \end{cases}.$$

Is the sequence of functions $(f_n)_{n \in \mathbb{N}}$ a Cauchy sequence in $C[0,1]$? Make a claim and prove that your assertion is correct.

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4. Consider the function $\varphi : \mathbb{R} \to \mathbb{R}$ defined by $\varphi(t) = \sqrt{n}$ if $t \in [n, n + \frac{1}{n^{5/2}}]$ for $n \in \mathbb{N}$ and $\varphi(t) = 0$ otherwise.

   (a) Show that φ is an $\mathcal{M}$-measurable function.

   (b) Does $\varphi \in L^1(\boldsymbol{m})$? Justify your conclusion.

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5. Let $(X, \mathcal{A}, \mu)$ be a measure space and $(E_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$. Suppose $E_{n+1} \subseteq E_n$ holds for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} \mu(E_n) = 0$.

   (a) Let $E = \bigcap\limits_{n\in\mathbb{N}} E_n$. Prove that $\lim\limits_{n\to\infty} \chi_{E_n}(t) = \chi_E(t)$.

   (b) Suppose $g \in L^1(\mu)$. Prove that $\lim\limits_{n\to\infty} \int_{E_n} f \, d\mu = 0$.

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6. Let $\mathcal{K}$ be the subset of $[0, 1]$ constructed in an analogous manner as the Cantor ternary set $\mathcal{C}$ except that each of the open intervals removed at the nth stage has length $\dfrac{2}{5^n}$. Do all of the following:

   (a) provide an inductive construction of $\mathcal{K}$,

   (b) show that $\mathcal{K}$ is Lebesgue measurable, and

   (c) find the *exact* value of $\boldsymbol{m}(\mathcal{K})$.