Abstract Algebra Comprehensive Exam

August 20, 2016

You have three hours to complete this examination. You may use a scientific calculator, but no other references are allowed. Please complete all six problems. All questions are equally weighted. You should present the work you wish to be graded on the sheets provided (one sheet per problem). Partial credit will be given. Please try to present your work as legibly as possible. You may use any lemmas or theorems that you can correctly state, but you must show that all conditions of those theorems are satisfied to establish your arguments. The problems are not presented in any particular order of difficulty, so you should scan the entire list and begin with the ones that seem the easiest to you. Good luck!

Problem	Score (out of 10 points)
1	
2	
3	
4	
5	
6	

Name: ______________________________ **Final Score:** ______________

Problem 1. The two parts below are not necessarily related.

(a) Suppose G is a group and $H \leq G$. If the index of H in G is 2, prove H is normal.

(b) Prove or provide a counterexample: Suppose G is a group. If H is a normal subgroup of G and K is a normal subgroup of H, then K is a normal subgroup of G.

Problem 2. Suppose G is a group of order $255 = 3 \cdot 5 \cdot 17$.

(a) Prove that G has a normal Sylow 17-subgroup.

(b) Prove that one of the Sylow 3- or 5-subgroups is also normal.

(c) Use the previous part to conclude that G has a subgroup of order 15 (you need **not** show it is normal).

Problem 3. Find all non-isomorphic abelian groups of order 540 by determining invariant factors, then check the completeness of your list with elementary divisors.

Problem 4. Use Zorn's Lemma to prove: If R is a commutative ring with identity, then R has at least one maximal ideal.

Problem 5. Prove that every Euclidean domain is a principal ideal domain.

Problem 6. Let $\alpha = \sqrt{2} + \sqrt{5} \in \mathbb{R}$.

(a) Find the minimal polynomial $m_\alpha(x)$ of α over $\mathbb{Q}$.

(b) Let E be the splitting field of $m_\alpha(x)$ over $\mathbb{Q}$. It can be shown (you **don't** need to) that $E = \mathbb{Q}(\sqrt{2}+\sqrt{5})$. Find the (isomorphism class of) the Galois group of E over $\mathbb{Q}$.