Analysis Comprehensive Exam

Department of Mathematics
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Saturday, August 20, 2016

Instructions. Please **write your name at the top of every page** and look to the board for possible corrections.

No references are permitted during the exam. Solutions must be written legibly and neatly on separate sheet(s) of paper with your name and problem number at the top of each page. Solutions should be organized with all variables defined and include a clear statement of what you plan to show. Include phrases between strings of computation (e.g. equalities or inequalities) to explain how you are proceeding, and provide complete and clear reasons for all of your steps (e.g. no statement is "clear" or "obvious" unless it is a definition or given as an assumption, and do not only name a theorem, also show explicitly that its hypotheses are satisfied). All problems are equally weighted. You have three (3) hours to submit your solutions.

Problem	Score (out of 10 points)
1	
2	
3	
4	
5	
6	

Name: ______________________ **Final Score:** ______________

Name: ____________________

1. Let (X, d) be a metric space. Let $a \in X$ and $r_0 \geq 0$. Must the set $F = \{x \in X \mid d(a, x) \leq r_0\}$ be closed in X? If so, prove it. Otherwise, give a counterexample of a metric d and show that under that metric, F not closed.

Name: ______________________________

2. Let $C[0,1]$ be equipped with its usual (supremum) metric d; that is,

$$d(f,g) = \sup\{|f(x) - g(x)| : x \in [0,1]\}, \quad f, g \in C[0,1].$$

For each $n \in N$, let $f_n : [0,1] \to \mathbb{R}$ be defined by $f_n(x) = x^n$. Is the sequence of functions $(f_n)_{n=1}^{\infty}$ a Cauchy sequence in $C[0,1]$? Make a claim and prove it.

Name: ____________________________

3. Let (X, d) be a metric space and $Y \subseteq X$.

   (a) Prove that if Y is nowhere dense in X, then $X \backslash \overline{Y}$ is a dense open subset of X.

   (b) Let $(F_n)_{n=1}^{\infty}$ be a sequence of nowhere dense subsets of a complete metric space X. Prove that $\displaystyle\bigcup_{n=1}^{\infty} F_n$ has empty interior.

Name: ______________________________

4. Suppose f, g and h are bounded real valued functions on $[0,1]$ with $f \leq g \leq h$. If f and h are Riemann integrable with
$$\int_0^1 f(x)\,dx = \int_0^1 h(x)\,dx,$$
prove that g is Riemann integrable.

Name: ____________________

5. Let $\boldsymbol{m}$ denote Lebesgue measure on $\mathbb{R}$ and $E \subseteq \mathbb{R}$. If $f \geq 0$ on $\mathbb{R}$ and $\int_E f \, d\boldsymbol{m} = 0$, prove that $f = 0$ almost everywhere on E.

Name: ______________________________

6. (a) Let $(f_n)_{n=1}^{\infty}$ be a sequence of real-valued Lebesgue measurable functions on $[0,1]$. State the Lebesgue Dominated Convergence Theorem for the sequence $(f_n)_{n=1}^{\infty}$.

   (b) Use part 6a to evaluate

$$\lim_{n\to\infty} \int_0^1 \frac{1}{(1+\frac{x}{n})^n\, x^{1/n}}\, dx$$