Analysis Comprehensive Exam

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Friday, January 13, 2017

Instructions. Please **write your name at the top of every page** and look to the board for possible corrections.

No references are permitted during the exam. Solutions must be written legibly and neatly on separate sheet(s) of paper with your name and problem number at the top of each page. Solutions should be organized with all variables defined and include a clear statement of what you plan to show. Include phrases between strings of computation (e.g. equalities or inequalities) to explain how you are proceeding, and provide complete and clear reasons for all of your steps (e.g. no statement is "clear" or "obvious" unless it is a definition or given as an assumption, and do not only name a theorem, also show explicitly that its hypotheses are satisfied). All problems are equally weighted. You have three (3) hours to submit your solutions.

1. Let (X, d) be a metric space and let $x_0 \in X$. Denote by $B_r(x_0)$ the ball in X centered at x_0 of radius $r > 0$; that is
$$B_r(x_0) = \{x \in X \mid d(x, x_0) < r\}.$$
Suppose that $0 < \delta < \epsilon$. Use only definitions to prove that if y is a limit point of the set $B_\delta(x_0)$, then $y \in B_\epsilon(x_0)$.

2. Let $C[0,1]$ be equipped with its usual (supremum) metric d; that is,
$$d(f,g) = \sup\{|f(x) - g(x)| \,:\, x \in [0,1]\}, \quad f, g \in C[0,1].$$
For each $n \in N$, let $f_n : [0,1] \to \mathbb{R}$ be defined by $f_n(x) = \dfrac{x}{1+x^n}$ for $x \in [0,1]$. Is the sequence of functions $(f_n)_{n=1}^\infty$ a Cauchy sequence in $C[0,1]$? Make a claim and prove it.

3. Let X and Y be metric spaces. Suppose $f : X \to Y$ is continuous and $K \subseteq X$ is sequentially compact. Prove that $f(K)$ is sequentially compact using the definition of sequential compactness. No credit will be awarded for other methods.

4. (a) Write a *precise* statement of the Arzela-Ascoli Theorem.
   (b) Suppose $(f_n)_{n=1}^\infty$ is a sequence of differentiable functions on $\mathbb{R}$ such that $f_n(0) \le 1$ and $f_n'(x) \le 1$ for all x and all n. If $f(x) = \lim_{n\to\infty} f_n(x)$ exists for all x, use part 4a to show that f is continuous on $\mathbb{R}$.

5. Let $\boldsymbol{m}$ denote Lebesgue measure on $\mathbb{R}$ and $E \subseteq \mathbb{R}$. Suppose that E_1 and E_2 are Lebesgue measurable sets and they both have finite measure. If $E_1 \subseteq E \subseteq E_2$ and $\boldsymbol{m}(E_1) = \boldsymbol{m}(E_2)$. Prove that E is Lebesgue measurable.

6. (a) Let $(f_n)_{n=1}^\infty$ be a sequence of real-valued Lebesgue measurable functions on $\mathbb{R}$. State the Lebesgue Dominated Convergence Theorem for the sequence $(f_n)_{n=1}^\infty$.
   (b) Use part 6a to prove that
$$\lim_{n\to\infty} \int_{\mathbb{R}} \frac{n \sin(x/n)}{x(x^2+1)} dx$$
exists and find its value.