Analysis Comprehensive Exam

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Instructions. Please **write your name at the top of every page** and look to the board for possible corrections.

No references are permitted during the exam. Solutions must be written legibly and neatly on separate sheet(s) of paper with your name and problem number at the top of each page. Solutions should be organized with all variables defined and include a clear statement of what you plan to show. Include phrases between strings of computation (e.g. equalities or inequalities) to explain how you are proceeding, and provide complete and clear reasons for all of your steps (e.g. no statement is "clear" or "obvious" unless it is a definition or given as an assumption, and do not only name a theorem, also show explicitly that its hypotheses are satisfied). All problems are equally weighted. You have three (3) hours to submit your solutions.

1. Let X be a set and let $\tilde{d}$ denote the discrete metric on X. Which subsets of the metric space $(X, \tilde{d})$ are compact? Make a claim and prove it.

2. Let $C[0,1]$ be equipped with its usual (supremum) metric d; that is,
$$d(f,g) = \sup\{|f(x) - g(x)| \,:\, x \in [0,1]\}, \quad f, g \in C[0,1].$$
For each $n \in \mathbb{N}$, let $f_n : [0,1] \to \mathbb{R}$ be defined by
$$f_n(x) = \begin{cases} 3 - nx, & 0 \le x \le \frac{1}{n} \\ 2, & \frac{1}{n} < x \le 1 \end{cases}.$$
Is the sequence of functions $(f_n)_{n=1}^{\infty}$ a Cauchy sequence in $C[0,1]$? Make a claim and prove it.

3. Suppose that X is a non-empty, complete, and countable metric space. Use Baire Category Theorem only to prove *directly* that X has an isolated point. Include a precise statement of the theorem in your proof and indicate how it is being applied.

4. Let μ^* denote Lebesgue outer measure on $\mathbb{R}$ and let $\mathbb{Q}$ denote the set of rational numbers. Prove that $\mu^*([0,1] \setminus \mathbb{Q}) = 1$.

   Hint: You may use the fact $\mu^*(\mathbb{Q}) = 0$.

5. (a) State the Arzelà-Ascoli Theorem.

   (b) Let $(f_n)_{n=1}^{\infty}$ be a sequence of twice differentiable functions on $[0,1]$ such that $f_n(0) = f_n'(0) = 0$ for all n and $|f_n''(x)| \le 1$ for all $x \in [0,1]$. Prove that there is a subsequence of $(f_n)_{n=1}^{\infty}$ that converges uniformly on $[0,1]$.
   Hint: Use the Mean Value Theorem.

6. (a) State Lebesgue's Dominated Convergence Theorem.

   (b) Suppose $f : [0,1] \to \mathbb{R}$ is continuous. Prove that $\displaystyle\lim_{n\to\infty} \int_0^1 x^n\, f(x)\, dx = 0$.

   (c) Suppose $f : [0,1] \to \mathbb{R}$ is continuous. Prove that $\displaystyle\lim_{n\to\infty} n \cdot \int_0^1 x^n\, f(x)\, dx = f(1)$.