Analysis Comprehensive Exam

Department of Mathematics
Florida Gulf Coast University
Saturday, January 13, 2018

Instructions. Please **write your name at the top of every page** and look to the board for possible corrections.

No references are permitted during the exam. Solutions must be written legibly and neatly on separate sheet(s) of paper with your name and problem number at the top of each page. Solutions should be organized with all variables defined and include a clear statement of what you plan to show. Include phrases between strings of computation (e.g. equalities or inequalities) to explain how you are proceeding, and provide complete and clear reasons for all of your steps (e.g. no statement is "clear" or "obvious" unless it is a definition or given as an assumption, and do not only name a theorem, also show explicitly that its hypotheses are satisfied). All problems are equally weighted. You have three (3) hours to submit your solutions.

---

1. Let $C[0,1]$ be equipped with its usual (supremum) metric d; that is,

$$d(f,g) = \sup\{|f(x) - g(x)| \,:\, x \in [0,1]\}, \quad f, g \in C[0,1].$$

For each $n \in \mathbb{N}$, let $f_n : [0,1] \to \mathbb{R}$ be defined by $f_n(x) = \dfrac{1 - x^n}{1 + x^2}$. Is the sequence of functions $(f_n)_{n=1}^{\infty}$ a Cauchy sequence in $C[0,1]$? Make a claim and prove it.

2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Define the set E by

$$E = \{x \in \mathbb{R}^n \,:\, f(x) = 3 \text{ and } d(x, \mathbf{0}) \leq 1\},$$

where $\mathbf{0}$ denotes the zero vector in $\mathbb{R}^n$ and d denotes the standard Euclidean metric on $\mathbb{R}^n$. Prove that E is compact.

3. Let (X, d) be a metric space and $Y \subseteq X$.
   (a) Prove that if Y is nowhere dense in X, then $X \backslash \overline{Y}$ is a dense open subset of X.
   (b) Let $(F_n)_{n=1}^{\infty}$ be a sequence of nowhere dense subsets of a complete metric space X. Prove that $\displaystyle\bigcup_{n=1}^{\infty} F_n$ has empty interior.

4. (a) Show that any subset $A \subset \mathbb{R}$ consisting of one single point, i.e., $A = \{a\}$ for some $a \in \mathbb{R}$, has zero outer measure.
   (b) Use part **??** to prove that the set of all rational numbers $\mathbb{Q}$ has outer measure zero.

5. Consider the series $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{1 + 2^n x^3}$.
   (a) Show (in detail) that the series converges uniformly over $(1, \infty)$.
   (b) Is the sum $f(x)$ of the series differentiable over $(1, \infty)$? Justify your answer.

6. For each $n \in \mathbb{N}$, let $f_n(x) = \dfrac{nx}{1+n^2x^2}$ for $x \in [0,1]$.

   (a) Prove that the sequence $(f_n)_{n=1}^{\infty}$ does not converge uniformly on $[0,1]$.

   (b) Find $\lim\limits_{n\to\infty} \displaystyle\int_0^1 f_n(x)dx$ if it exists.