Syllabus for Comprehensive Analysis Exam

1. Definitions (in a metric space unless otherwise stated)

   (a) upper/lower bound of an ordered set

   (b) least upper (greatest lower) bound property

   (c) supremum/infimum of a set

   (d) Finite set, countable set

   (e) metric d on a set X, metric space

   (f) discrete metric, discrete metric space

   (g) ϵ-neighborhood centered at x_0, $N_\epsilon(x_0)$

   (h) interior points, limit points, isolated points

   (i) open set, closed set, closure

   (j) diameter of a set, bounded set

   (k) compact, open cover, subcover

   (l) convergent sequence, limit, divergent sequence,

   (m) subsequence, bounded sequence

   (n) Cauchy sequence

   (o) complete metric space

   (p) dense (two characterizations, nowhere dense

   (q) first category, second category

   (r) sequentially compact, totally bounded

   (s) continuous function (at a point, on a set)

   (t) bounded function

2. Examples

   (a) Metric spaces: R^n, $C[a, b]$ (with the usual supremum metric and the "L1/integral" metric)

   (b) $(X, \tilde{d})$, where $\tilde{d}$ denotes the discrete metric

   (c) Open neighborhoods, limit points, closure of sets in a discrete metric space

   (d) Non-convergent Cauchy sequence

   (e) Metrics for which $\mathbb{R}, \mathbb{Q}, C[a, b]$, a subset of $\mathbb{R}$ are complete or not complete

   (f) Importance of completeness in Baire's Theorem

   (g) Compact subsets of $\mathbb{R}$ and subsets of $\mathbb{R}$ that are not compact

   (h) Non-compactness of the closed unit ball in $C[0, 1]$

   (i) Continuous functions from $C[a, b]$ to $\mathbb{R}$.

3. Theorems (in a metric space unless otherwise stated):

   (a) Properties of open sets and closed sets

   (b) Characterizations of closed sets

   (c) Properties of convergent sequences and Cauchy sequences

   (d) Properties of subsets of complete metric spaces

   (e) Characterization of dense sets

   (f) Baire's Theorem

   (g) Baire Category Theorem

   (h) Properties of nowhere dense sets

   (i) Characterizations of compactness

   (j) Properties of compact sets

   (k) Archimedean property of $\mathbb{R}$

   (l) Density of $\mathbb{Q}$ in $\mathbb{R}$

   (m) Monotone Convergence Theorem ($\mathbb{R}$)

   (n) Bolzano Weierstrass Theorem ($\mathbb{R}$)

(o) Cauchy Criterion ($\mathbb{R}$)
(p) Heine-Borel Theorem ($\mathbb{R}^n$)
(q) Characterizations of continuity ($\epsilon - \delta$ def., sequential, topological)
(r) Criterion for Discontinuity
(s) Preservation of Compactness
(t) Extreme Value Theorem

4. Riemann integral
   (a) Development of the Riemann integral via upper and lower sum
   (b) upper and lower integrals
   (c) Integrability criteria
   (d) Fundamental Theorem of Calculus and Mean Value Theorem
   (e) Basic properties of Riemann integral
5. Sequences and Series of Functions
   (a) Pointwise convergence
   (b) Uniform convergence
   (c) Absolute convergence
   (d) Cauchy criterion
   (e) Term-by-term integration
   (f) Term-by-term differentiation
   (g) Weierstrass M-Test
   (h) Equicontinuity and Ascoli-Arzela Theorem
6. Measures
   (a) Definition of outer measure and properties;
   (b) Measurable sets
   (c) Lebesgue meausre
   (d) Elementary properties of measures
7. The Lebesgue integral
   (a) Measurable functions
   (b) Elementary properties of measurable functions
   (c) Simple functions
   (d) Approximation of measurable functions by simple functions
   (e) Definition of the integral with respect to a measure
   (f) Properties of Lebesgue integral
   (g) Bounded convergence theorem
   (h) Monotone convergence theorem, Fatou's Lemma, Lebesgue Dominated Convergence Theorem
   (i) Relationship between Riemann and Lebesgue integrals, and Lebesgue's Riemann-Integrability Criterion

Last update: June 2017