COMPREHENSIVE EXAMINATION - MODERN ALGEBRA - AUGUST 2018

You have **three hours** to complete this examination which consists of **six** (**6**) equally weighted problems. No references or calculators are permitted. You should present the work you wish to be graded on the scratch sheets provided (one problem per sheet). Partial credit will be given. Please try to present your work as legibly as possible. You may use any lemmas or theorems that you can correctly state, but you must demonstrate their applicability to your arguments. The problems are not presented in any particular order of difficulty, so you should scan the entire list and begin with the ones that seem the easiest to you. Try to allocate your time to maximize points. Good luck!

1) State and prove in detail Lagrange's theorem on the possible orders of a subgroup.

2) State and prove the First Isomorphism Theorem for groups.

3) Show that a group of order 380 cannot be simple.

4) Prove that an algebraically closed field must be infinite.

5) Find the splitting field F over $\mathbb{Q}$ for $p(x) = (x^2 - 3)(x^2 + 2)$ and:

(i) Determine the galois group $Gal(F/\mathbb{Q})$.

(ii) Draw the lattice of subfields of F and the corresponding lattice of subgroups of $Gal(F/\mathbb{Q})$.

(iii) State in complete detail the theorem that relates the lattices in (ii)

6) (i) Show that $p(x) = x^3 + 3x + 6$ is irreducible over $\mathbb{Q}$.

(ii) If θ is a zero of $p(x)$, find $(1 + \theta)^{-1} \in \mathbb{Q}(\theta)$ Hint: you may need to solve a small linear system