Analysis Comprehensive Exam

Department of Mathematics
Florida Gulf Coast University
Saturday, August 25, 2018

Instructions. Please **write your name at the top of every page** and look to the board for possible corrections.

No references are permitted during the exam. Solutions must be written legibly and neatly on separate sheet(s) of paper with your name and problem number at the top of each page. Solutions should be organized with all variables defined and include a clear statement of what you plan to show. Include phrases between strings of computation (e.g. equalities or inequalities) to explain how you are proceeding, and provide complete and clear reasons for all of your steps (e.g. no statement is "clear" or "obvious" unless it is a definition or given as an assumption, and do not only name a theorem, also show explicitly that its hypotheses are satisfied). All problems are equally weighted. You have three (3) hours to submit your solutions.

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1. Let K_j, $j = 1, 2, \ldots$ be compact sets in a metric space. Give a short proof or counterexample to each of the following assertions.
   (a) $K_1 \cap K_2$ is compact.
   (b) $K_1 \cup K_2$ is compact.
   (c) $\displaystyle\bigcup_{j=1}^{\infty} K_j$ is compact.

2. Let $C[0,1]$ be equipped with its usual (supremum) metric d; that is,
$$d(f,g) = \sup\{|f(x) - g(x)| : x \in [0,1]\}, \quad f, g \in C[0,1].$$
For each $n \in \mathbb{N}$, let $f_n : [0,1] \to \mathbb{R}$ be defined by
$$f_n(x) = \frac{2 - x^2}{1 + x^n} \quad \text{for } x \in [0,1].$$
Is the sequence of functions $\{f_n\}$ a Cauchy sequence in $C[0,1]$? Make a claim and prove it.

3. (a) State the definition of complete metric space.
   (b) Let (X, d) be a metric space with a dense subset $A \subset X$ such that every Cauchy sequence in A converges in X. Prove that X is complete.

4. Let μ^* denote Lebesgue outer measure on $\mathbb{R}$. Show that if E_1 and E_2 are Lebesgue measurable subsets of $\mathbb{R}$, then
$$\mu^*(E_1 \cup E_2) + \mu^*(E_1 \cap E_2) = \mu^*(E_1) + \mu^*(E_2).$$

5. Consider the series $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^{5/2}}$.
   (a) Is $f(x)$ continuous on $\mathbb{R}$? Justify your answer.
   (b) Is $f(x)$ differentiable on $\mathbb{R}$? Justify your answer.

6. (a) Let $\{f_n\}$ be a sequence of real-valued Lebesgue measurable functions on a measurable set E. State Lebesgue Dominated Convergence Theorem for the sequence $\{f_n\}$.

   (b) Use part 6a to find $\lim_{n\to\infty} \int_1^\infty \frac{\ln(1+nx)}{1+x^4\ln(n)}dx$ and justify your answer.

   Hint: Use the fact that $\ln(x)$ is monotone increasing.