Syllabus for Comprehensive Analysis Exam

1. Definitions (in a metric space unless otherwise stated)
   (a) supremum/infimum of a set $A \subseteq \mathbb{R}$
   (b) metric d on a set X, metric space
   (c) discrete metric space
   (d) ϵ (open) ball centered at x_0
   (e) closed ball centered at x_0 of radius r
   (f) sphere centered at x_0 of radius ϵ
   (g) open set, closed set, closure
   (h) diameter of a set, bounded set
   (i) interior points, limit points, isolated points
   (j) convergent sequence, limit, divergent sequence, subsequence
   (k) Cauchy sequence
   (l) complete metric space
   (m) dense, nowhere dense
   (n) compact, sequentially compact, open cover, subcover, totally bounded
   (o) equicontinuous subset of $C(K)$
   (p) continuous function
   (q) vector space, norm
   (r) normed vector space
   (s) ~~metric induced by a norm~~
   (t) ~~Banach space~~
   (u) ~~subspace, linear combination, span~~
   (v) ~~linearly independent/dependent set, dimension, (Hamel) basis~~
   (w) ~~finite dimensional vector space~~
   (x) ~~convergent sequence in a normed vector space~~
   (y) ~~Cauchy sequence in a normed vector space~~
   (z) ~~equivalent norms~~

2. Examples
   (a) Metric spaces: R^n, $\mathbb{C}^n$, ℓ^∞, ℓ^p for $p \in [1, \infty)$
   (b) $C[a, b]$ (with the usual supremum metric and the "L^1/integral" metric)
   (c) $(X, \tilde{d})$, where $\tilde{d}$ denotes the discrete metric
   (d) Open balls, limit points, closure of sets in a discrete metric space
   (e) Non-convergent Cauchy sequence
   (f) Metrics for which $\mathbb{R}, \mathbb{Q}, C[a, b]$, a subset of $\mathbb{R}$ are complete or not complete
   (g) Importance of completeness in Baire's Theorem
   (h) Compact subsets of $\mathbb{R}$ and subsets of $\mathbb{R}$ that are not compact
   (i) Non-compactness of the closed unit ball in $C[0, 1]$
   (j) Continuous functions from $C[0, 1]$ to $\mathbb{R}$.
   (k) ~~Normed vector spaces and Banach spaces~~
   (l) ~~A metric not induced by a norm~~

3. Theorems (in a metric space unless otherwise stated):
   (a) Properties of open sets and closed sets
   (b) Characterizations of closed sets
   (c) Properites of convegent sequences and Cauchy sequences
   (d) Properties of subsets of complete metric spaces
   (e) Charaterization of dense sets
   (f) Baire's Theorems
   (g) Properties of nowhere dense sets
   (h) Characterizations of compactness ("Main theorem for compactness")
   (i) Properties of compact sets

(j) Bolzano Weierstrass Theorem ($\mathbb{R}$)
(k) Heine-Borel Theorem ($\mathbb{R}^n$)
(l) Arzela-Ascoli Theorem ($C(K)$)
(m) Characterizations of continuity
(n) ~~Preservation of Compactness~~
(o) Extreme Value Theorem
(p) ~~Properties of norms on vector spaces~~
(q) ~~Equivalence of Norms Theorem~~
(r) ~~Corollaries to Equivalence of Norms Theorem~~

4. Riemann integral
   (a) Development of the Riemann integral via upper and lower sum
   (b) upper and lower integrals
   (c) Integrability criteria
   (d) Fundamental Theorem of Calculus and Mean Value Theorem
   (e) Basic properties of Riemann integral

5. Sequences and Series of Functions
   (a) Pointwise convergence
   (b) Uniform convergence
   (c) Absolute convergence
   (d) Cauchy criterion
   (e) Term-by-term integration
   (f) Term-by-term differentiation
   (g) Weierstrass M-Test
   (h) Equicontinuity and Ascoli-Arzela Theorem

6. Measures
   (a) Definition of outer measure and properties;
   (b) Measurable sets
   (c) Lebesgue meausre
   (d) Elementary properties of measures

7. The Lebesgue integral
   (a) Measurable functions
   (b) Elementary properties of measurable functions
   (c) Simple functions
   (d) Approximation of measurable functions by simple functions
   (e) Definition of the integral with respect to a measure
   (f) Properties of Lebesgue integral
   (g) Bounded convergence theorem
   (h) Monotone convergence theorem, Fatou's Lemma, Lebesgue Dominated Convergence Theorem
   (i) Relationship between Riemann and Lebesgue integrals, and Lebesgue's Riemann-Integrability Criterion

Last update: June 2018