Approximation by meromorphic matrix-valued functions

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Our plan

1. Nehari-Takagi problem for scalar-valued functions
   1. Generalization of Classical Interpolation Problems
   2. Operator Theory

2. Matrix-valued functions: Are there suitable analogs of all previous results?
The case of scalar-valued functions
A class of meromorphic functions

Throughout,

1. $L^\infty$ is the space of bounded functions on the unit circle $\mathbb{T}$;

2. $H^\infty$ is the Hardy space of the unit disc $\mathbb{D}$ consisting of $L^\infty$ functions whose Fourier coefficients of negative index vanish;

3. for $k \geq 0$, $B_k$ is the set of Blaschke products of degree at most $k$, i.e. $b \in B_k$ if and only if $b(\zeta) = c \prod_{j=1}^{d} \frac{\zeta - \lambda_j}{1 - \bar{\lambda}_j \zeta}$ where $d \leq k$, $\lambda_j \in \mathbb{D}$, and $c \in \mathbb{T}$; and

4. for $k \geq 0$, $H^\infty_{(k)} \overset{\text{def}}{=} B_k^{-1} H^\infty$ is the set of meromorphic functions with at most $k$ poles in the unit disc $\mathbb{D}$ (counting multiplicities) which are bounded near the unit circle $\mathbb{T}$. 
Let $\varphi \in L^\infty$. The **Nehari-Takagi problem** is to find a $q \in H^\infty_\(k\)$ which is closest to $\varphi$ with respect to the $L^\infty$-norm, i.e. to find $q \in H^\infty_\(k\)$ such that

$$\|\varphi - q\|_\infty = \text{dist}_{L^\infty}(\varphi, H^\infty_\(k\)) = \inf_{f \in H^\infty_\(k\)} \|\varphi - f\|_\infty.$$  

Any such function $q$ is called a **best approximant in $H^\infty_\(k\)$ to $\varphi$**.

**Question 1.** *Existence?* Yes.

**Question 2.** *Uniqueness?* Not always, but a sufficient condition is continuity of $\varphi$ on $\mathbb{T}$.
Who cares?

1. Mathematicians who enjoy operator theoretic complex function theory
   - Generalization of Classical Interpolation Problems
   - Connections with Operator Theory

2. Engineers interested in $H^\infty$-control and signal theory
   - Systems can be described in the “frequency domain” as multiplication operator by a “transfer function” which belongs to certain Hardy classes if the system has certain stability properties
   - Rational functions are the “transfer functions” of systems having finite-dimensional state-space (and so these can be handled in practice)
Given \( c_0, c_1, \ldots, c_n \in \mathbb{C} \), the CF-problem is to find out when there is an \( f \in H^\infty \) such that

\[
\hat{f}(j) = c_j, \quad 0 \leq j \leq n, \quad \text{and} \quad \|f\|_{\infty} \leq 1.
\]

Consider the function \( g = c_0 + c_1 z + \ldots + c_n z^n \). Then any function whose Taylor coefficients are \( c_0, c_1, \ldots, c_n \) is of the form \( g + z^{n+1} h \) for some \( h \in H^\infty \). Therefore, the CF-problem is solvable iff

\[
1 \geq \inf\{\|g + z^{n+1} h\|_{\infty} : h \in H^\infty\} = \text{dist}_{L^\infty}(\bar{z}^{n+1} g).
\]
Nevanlinna-Pick (NP) Interpolation

Given distinct points \( \lambda_i \in \mathbb{D}, 1 \leq i \leq n \), and points \( w_i \in \mathbb{C}, 1 \leq i \leq n \), the NP-problem is to find out when there is an \( f \in H^\infty \) such that

\[
f(\lambda_i) = w_i, \ 1 \leq i \leq n, \ \text{and} \ \|f\|_\infty \leq 1.
\]

Consider the functions

\[
B_i = \prod_{j \neq i} \frac{z - \lambda_j}{1 - \bar{\lambda}_j z}, 1 \leq i \leq n, \ \text{and} \ g = \sum_{i=1}^n \frac{w_i}{B_i(\lambda_i)} B_i.
\]

Then any function that interpolates \( w_i \) at \( \lambda_i \) is of the form \( g - Bh \) for some \( h \in H^\infty \), where \( B \) a Blaschke product whose zeros are precisely \( \lambda_1, \ldots, \lambda_n \). Thus, the NP-problem has a solution iff

\[
1 \geq \inf \{ \|g - Bh\| : h \in H^\infty \} = \operatorname{dist}_{L^\infty}(\bar{B}g, H^\infty).
\]
For $f \in L^2$, let $\mathbb{P}_+$ and $\mathbb{P}_-$ denote the orthogonal projections onto $H^2 = \{ f \in L^2 : \hat{f}(n) = 0 \text{ for } n < 0 \}$ and $H^- = L^2 \ominus H^2$, i.e.

$$\mathbb{P}_+ f = \sum_{n \geq 0} \hat{f}(n) z^n \quad \text{and} \quad \mathbb{P}_- f = \sum_{n < 0} \hat{f}(n) z^n.$$

Given $\varphi \in L^\infty$, we define the Hankel operator $H_\varphi$ and the Toeplitz operator $T_\varphi$ by

$$H_\varphi f = \mathbb{P}_- \varphi f \quad \text{and} \quad T_\varphi f = \mathbb{P}_+ \varphi f,$$

$f \in H^2$,

respectively.
Some classical results

1. Kronecker (1881): A Hankel operator $H_q$ has finite rank if and only if $q \in H^\infty(k)$. Consequently, $\mathbb{P}_-q$ is a rational function whose poles lie in $\mathbb{D}$ and rank $H_q = \deg \mathbb{P}_-q$.

2. Nehari (1957): $\|H_\varphi\| = \text{dist}_{L^\infty}(\varphi, H^\infty)$

3. Hartman (1958): $\|H_\varphi\|_e = \text{dist}_{L^\infty}(\varphi, H^\infty + C)$, where $\|T\|_e$ denotes the essential norm of the operator $T$, i.e.

$$\|T\|_e \overset{\text{def}}{=} \inf \{ \|T - K\| : K \text{ is compact} \}.$$

4. Adamyan, Arov, Krein (1970): Existence of a best approximants in $H^\infty(k)$ and $s_k(H_\varphi) = \text{dist}_{L^\infty}(\varphi, H^\infty(k))$, where $s_k(T)$ denotes the $k$th singular value of the operator $T$, i.e.

$$s_k(T) \overset{\text{def}}{=} \inf \{ \|T - R\| : \text{rank } R \leq k \}.$$
A function $\varphi \in L^\infty$ is called $k$-admissible if $\|H\varphi\|_e < s_k(H\varphi)$ holds. E.g. functions in $C \setminus H^\infty_{(k)}$ are admissible.

**Theorem (Peller (1990))**

Let $\varphi$ be $k$-admissible. Then $q$ is the unique best meromorphic approximant in $H^\infty_{(k)}$ to $\varphi$ iff

1. $\varphi - q$ has constant modulus on $\mathbb{T}$, and
2. the Toeplitz operator $T_{\varphi - q}$ is Fredholm and has index

$$\text{ind } T_{\varphi - q} = 2k + \mu,$$

where $\mu$ denotes the multiplicity of the singular value $s_k(H\varphi)$ of the Hankel operator $H\varphi$.

Why is this characterization useful?
Theorem (Peller (1990))

Let $\varphi$ be $k$-admissible and $q$ be the best approx. in $H_{(k)}^\infty$ to $\varphi$. Suppose $s$ is a singular value of $H_\varphi$ of multiplicity $\mu \geq 2$ and

$$s = s_k(H_\varphi) = s_{k+1}(H_\varphi) = \ldots = s_{k+\mu-1}(H_\varphi) > s_{k+\mu}(H_\varphi).$$

Let $\lambda \in \mathbb{D}$, $b = \frac{z - \lambda}{1 - \lambda z}$, and $\psi = \overline{b}\varphi$. Then the following hold:

1. If $\lambda$ is a zero of $q$, then

$$s = s_k(H_\psi) = s_{k+1}(H_\psi) = \ldots = s_{k+\mu}(H_\psi) > s_{k+\mu+1}(H_\psi).$$

2. If $\lambda$ is not a zero of $q$, then

$$s = s_{k+1}(H_\psi) = s_{k+2}(H_\psi) = \ldots = s_{k+\mu-1}(H_\psi) > s_{k+\mu}(H_\psi).$$
### Theorem (Peller (1990))

Let $\varphi$ be $k$-admissible and $q$ be the best approx. in $H_\infty^{(k)}$ to $\varphi$. Suppose $s$ is a singular value of $H_\varphi$ of multiplicity $\mu \geq 2$ and

$$s = s_k(H_\varphi) = s_{k+1}(H_\varphi) = \ldots = s_{k+\mu-1}(H_\varphi) > s_{k+\mu}(H_\varphi).$$

Let $\lambda \in \mathbb{D}$, $b = \frac{z-\lambda}{1-\lambda z}$, and $\psi = b\varphi$. Then the following hold:

1. **If $\lambda$ is a pole of $q$, then**
   $$s = s_{k-1}(H_\psi) = s_k(H_\psi) = \ldots = s_{k+\mu-1}(H_\psi) > s_{k+\mu}(H_\psi).$$

2. **If $\lambda$ is not a pole of $q$, then**
   $$s = s_{k+1}(H_\psi) = s_{k+2}(H_\psi) = \ldots = s_{k+\mu-2}(H_\psi) > s_{k+\mu-1}(H_\psi).$$
The degree of the best meromorphic approximant

Theorem (Peller-Khrushčëv (1982))

If \( \varphi \) is a rational function with poles outside \( \mathbb{T} \) of degree, then the best meromorphic approximant \( q \) in \( H^\infty_{(k)} \) to \( \varphi \) is also rational and

\[
\deg q \leq \deg \, \varphi - 1 \quad \text{unless} \quad \varphi \in H^\infty_{(k)}.
\]
The case of matrix-valued functions

Why should we do this?

In systems theory,

1. scalar-valued functions correspond to single input - single output systems
2. matrix-valued functions correspond to multiple input - multiple output systems
**Notation**

1. $\mathbb{M}_n$ denotes the space of $n \times n$ matrices equipped with the operator norm $\| \cdot \|_{\mathbb{M}_n}$.

2. $L^\infty(\mathbb{M}_n)$ is equipped with $\| \Phi \|_\infty = \text{ess sup} \| \Phi(\zeta) \|_{\mathbb{M}_n}$.

3. $H^\infty_{(k)}(\mathbb{M}_n)$ consists of matrix-valued functions $Q$ with at most $k$ poles in $\mathbb{D}$.

4. $Q \in L^\infty(\mathbb{M}_n)$ is said to have **at most** $k$ poles in $\mathbb{D}$ if there is a Blaschke-Potapov product $B$ of degree $k$ such that $QB \in H^\infty(\mathbb{M}_n)$. 
What is a Blaschke-Potapov product?

A finite **Blaschke-Potapov product** is an $n \times n$ matrix-valued function of the form $B = UB_1 \ldots B_m$, where

$$B_i(z) = \frac{z - \lambda_i}{1 - \overline{\lambda}_i z} P_i + (I - P_i)$$

with $\lambda_i \in \mathbb{D}$ and orthogonal projection $P_i$ on $\mathbb{C}^n$, $1 \leq i \leq m$, and $U$ is a unitary matrix. We define $\deg B = \sum_{i=1}^{m} \text{rank } P_i$.

**Why count poles in this way?**

In nicer language:

$$B(z) = U_0 \left( \begin{array}{cc}
\frac{z-a_1}{1-\overline{a}_1 z} & 0 \\
0 & I_{n-1}
\end{array} \right) U_1 \ldots U_{k-1} \left( \begin{array}{cc}
\frac{z-a_k}{1-\overline{a}_k z} & 0 \\
0 & I_{n-1}
\end{array} \right) U_k,$$

where $a_1, \ldots, a_k \in \mathbb{D}$ and $U_0, U_1, \ldots, U_k$ are constant $n \times n$ unitary matrices.
Nehari-Takagi problem

Definition

Given $\Phi \in L^\infty(M_n)$, we say that $Q$ is a best approximation in $H^\infty_k(M_n)$ to $\Phi$ if $Q$ has at most $k$ poles and

$$\|\Phi - Q\|_{L^\infty(M_n)} = \text{dist}_{L^\infty(M_n)}(\Phi, H^\infty_k(M_n)).$$

As before, given $\Phi \in L^\infty(M_n)$, we define

1. the Toeplitz operator $T_\Phi : H^2(\mathbb{C}^n) \to H^2(\mathbb{C}^n)$ by

$$T_\Phi f = \mathbb{P}_+ \Phi f \quad \text{for } f \in H^2(\mathbb{C}^n),$$

and

2. the Hankel operator $H_\Phi : H^2(\mathbb{C}^n) \to H_-^2(\mathbb{C}^n)$ by

$$H_\Phi f = \mathbb{P}_- \Phi f \quad \text{for } f \in H^2(\mathbb{C}^n).$$
(Known to specialists) A Hankel operator $H_Q$ has finite rank if and only if $Q \in H^\infty_{(k)}$. Consequently, $P_q$ is a rational function whose poles lie in $\mathbb{D}$ and rank $H_q = \deg P_q$.

Page (1970): $\|H_P\| = \text{dist}_{L^\infty}(P, H^\infty(M_n))$

Ball-Helton (1983), Treil (1986): Existence of a best approximants in $H^\infty_{(k)}$ and $s_k(H_\varphi) = \text{dist}_{L^\infty}(\varphi, H^\infty_{(k)})$, where $s_k(T)$ denotes the $k$th singular value of the operator $T$, i.e.

$$ s_k(T) \overset{\text{def}}{=} \inf \{ \| T - R \| : \text{rank} \ R \leq k \}. $$

Treil (1986): $\|H_\varphi\|_e = \text{dist}_{L^\infty}(\varphi, H^\infty + C)$, where $\| T \|_e$ denotes the essential norm of the operator $T$, i.e.

$$ \| T \|_e \overset{\text{def}}{=} \inf \{ \| T - K \| : K \text{ is compact} \}. $$

How about uniqueness of a best meromorphic approximant in $H^\infty_{(k)}(\mathbb{M}_n)$?
Definition (Young)

Let $k \geq 0$ and $\Phi \in L^\infty(\mathbb{M}_n)$. We say that $Q$ is a superoptimal meromorphic approximant of $\Phi$ in $H^{\infty}_{(k)}(\mathbb{M}_n)$ if $Q$ has at most $k$ poles in $\mathbb{D}$ and minimizes the essential suprema of singular values $s_j((\Phi - Q)(\zeta))$, $j \geq 0$, with respect to the lexicographic ordering:

$$Q \text{ minimizes } \text{ess sup}_{\zeta \in T} s_0(\Phi(\zeta) - Q(\zeta)) \text{ on } H^{\infty}_{(k)}(\mathbb{M}_n)$$

then...
$$\text{then } \text{minimize } \text{ess sup}_{\zeta \in T} s_1(\Phi(\zeta) - Q(\zeta))$$

then...
$$\text{then } \text{minimize } \text{ess sup}_{\zeta \in T} s_2(\Phi(\zeta) - Q(\zeta)) \ldots \text{ and so on.}$$

For $j \geq 0$, the number $t^{(k)}_j \overset{\text{def}}{=} \text{ess sup}_{\zeta \in T} s_j(\Phi(\zeta) - Q(\zeta))$ is called the $j$th superoptimal singular value of $\Phi$ of degree $k$. 
Superoptimal meromorphic approximation in $H^\infty_k(M_n)$

**Definition (Young)**

Let $k \geq 0$ and $\Phi \in L^\infty(M_n)$. We say that $Q$ is a superoptimal meromorphic approximant of $\Phi$ in $H^\infty_k(M_n)$ if $Q$ has at most $k$ poles in $\mathbb{D}$ and minimizes the essential suprema of singular values $s_j((\Phi - Q)(\zeta))$, $j \geq 0$, with respect to the lexicographic ordering:

\[
Q \text{ minimizes } \operatorname{ess sup}_{\zeta \in \mathbb{T}} s_0(\Phi(\zeta) - Q(\zeta)) \text{ on } H^\infty_k(M_n)
\]

then...

\[
\text{ then... minimize } \operatorname{ess sup}_{\zeta \in \mathbb{T}} s_1(\Phi(\zeta) - Q(\zeta))
\]

then...

\[
\text{ then... minimize } \operatorname{ess sup}_{\zeta \in \mathbb{T}} s_2(\Phi(\zeta) - Q(\zeta)) \ldots \text{ and so on.}
\]

For $j \geq 0$, the number $t_j^{(k)} \overset{\text{def}}{=} \operatorname{ess sup}_{\zeta \in \mathbb{T}} s_j(\Phi(\zeta) - Q(\zeta))$ is called the $j$th superoptimal singular value of $\Phi$ of degree $k$. 
Definition (Young)

Let $k \geq 0$ and $\Phi \in L^\infty(\mathbb{M}_n)$. We say that $Q$ is a superoptimal meromorphic approximant of $\Phi$ in $H^{\infty}_{(k)}(\mathbb{M}_n)$ if $Q$ has at most $k$ poles in $\mathbb{D}$ and minimizes the essential suprema of singular values $s_j((\Phi - Q)(\zeta))$, $j \geq 0$, with respect to the lexicographic ordering:

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then... minimize $\text{ess sup}_{\zeta \in \mathbb{T}} s_2(\Phi(\zeta) - Q(\zeta))$... and so on.

For $j \geq 0$, the number $t_j^{(k)} \overset{\text{def}}{=} \text{ess sup}_{\zeta \in \mathbb{T}} s_j(\Phi(\zeta) - Q(\zeta))$ is called the $j$th superoptimal singular value of $\Phi$ of degree $k$. 

We say that $\Phi$ is $k$-admissible if $\|H_\Phi\|_e$ is less than the smallest non-zero superoptimal singular value of $\Phi$ of degree $k$ and $s_k(H_\Phi) < s_{k-1}(H_\Phi)$.


*If $\Phi$ is $k$-admissible, then $\Phi$ has a unique superoptimal meromorphic approximant in $H_\infty^{(k)}(\mathbb{M}_n)$ and $s_j(\Phi(\zeta) - Q(\zeta)) = t_j^{(k)}$ for a.e. $\zeta \in \mathbb{T}$, $j \geq 0$.***
Theorem (A.C. 2012)

Suppose

1. \( \Phi \) is \( k \)-admissible and
2. \( \Phi \) has \( n \) non-zero superoptimal singular values of degree \( k \).

Then the Toeplitz operator \( T_{\Phi - Q} \) is Fredholm and

\[
\text{ind } T_{\Phi - Q} = \dim \ker T_{\Phi - Q} > 0.
\]
Theorem (A.C. 2012)

Suppose

1. $\Phi$ is $k$-admissible and
2. $\Phi$ has $n$ non-zero superoptimal singular values of degree $k$.

Then the Toeplitz operator $T_{\Phi - Q}$ is Fredholm and

$$\text{ind } T_{\Phi - Q} = \dim \ker T_{\Phi - Q} > 0.$$
Let $\Psi = \Phi - Q$ and $W = \Psi^*\Psi$. Then

1. $\ker T_\Psi = \{ f \in H^2(\mathbb{C}^n) : \| H_\Psi f \|_2 = \| \Psi \|_2 \}$

2. $W$ is invertible a.e. on $\mathbb{T}$ and

\[
\| W(\zeta)^{-1} \| = s_{n-1}^{-1}(W(\zeta)) = t_{n-1}^{-2} \text{ for a.e. } \zeta \in \mathbb{T}
\]

3. $W^{1/2} \ker T_\Psi = W^{1/2}\{ f \in H^2(\mathbb{C}^n) : \| H_\Psi f \|_2 = \| \Psi f \|_2 \} = W^{1/2}\{ f \in H^2(\mathbb{C}^n) : \| H_\Psi f \|_2 = \| W^{1/2}f \|_2 \} = \{ \xi \in W^{1/2}H^2(\mathbb{C}^n) : \| H_\Psi W^{-1/2}\xi \|_2 = \| \xi \|_2 \}$

**Conclusion:** The operator $H_\Psi W^{-1/2}$ defined on $W^{1/2}H^2(\mathbb{C}^n)$ and equipped with the $L^2$-norm has norm equal to 1.

4. $\| H_\Psi W^{-1/2}|W^{1/2}H^2(\mathbb{C}^n)\|_e < 1$

Hence, the space of maximizing vectors of $H_\Psi W^{-1/2}$ is finite dimensional.
Let $\Psi = \Phi - Q$ and $W = \Psi^*\Psi$. Then

1. $\ker T_\Psi = \{ f \in H^2(\mathbb{C}^n) : \| H_\Psi f \|_2 = \| \Psi f \|_2 \}$

2. $W$ is invertible a.e. on $\mathbb{T}$ and
$$\| W(\zeta)^{-1} \| = s_{n-1}^{-1}(W(\zeta)) = t_{n-1}^{-2}$$
for a.e. $\zeta \in \mathbb{T}$

3. $W^{1/2} \ker T_\Psi = W^{1/2}\{ f \in H^2(\mathbb{C}^n) : \| H_\Psi f \|_2 = \| \Psi f \|_2 \}$
   $$= W^{1/2}\{ f \in H^2(\mathbb{C}^n) : \| H_\Psi f \|_2 = \| W^{1/2} f \|_2 \}$$
   $$= \{ \xi \in W^{1/2} H^2(\mathbb{C}^n) : \| H_\Psi W^{-1/2} \xi \|_2 = \| \xi \|_2 \}$$

Conclusion: The operator $H_\Psi W^{-1/2}$ defined on $W^{1/2} H^2(\mathbb{C}^n)$ and equipped with the $L^2$-norm has norm equal to 1.

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\]
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Hence, the space of maximizing vectors of $H_\Psi W^{-1/2}$ is finite dimensional.
A taste of the proof of \( \text{dim ker } T_{\Phi - Q} < \infty \)

Let \( \Psi = \Phi - Q \) and \( W = \Psi^*\Psi \). Then

1. \( \ker T_{\Psi} = \{ f \in H^2(\mathbb{C}^n) : \| H_{\Psi}f \|_2 = \| \Psi \|_2 \} \)

2. \( W \) is invertible a.e. on \( \mathbb{T} \) and
   \[ \| W(\zeta)^{-1} \| = s_{n-1}^{-1}(W(\zeta)) = t_{n-1}^{-2} \] for a.e. \( \zeta \in \mathbb{T} \)

3. \( W^{1/2} \ker T_{\Psi} = W^{1/2} \{ f \in H^2(\mathbb{C}^n) : \| H_{\Psi}f \|_2 = \| \Psi f \|_2 \} \)
   \[ = W^{1/2} \{ f \in H^2(\mathbb{C}^n) : \| H_{\Psi}f \|_2 = \| W^{1/2}f \|_2 \} \]
   \[ = \{ \xi \in W^{1/2}H^2(\mathbb{C}^n) : \| H_{\Psi}W^{-1/2}\xi \|_2 = \| \xi \|_2 \} \]

Conclusion: The operator \( H_{\Psi}W^{-1/2} \) defined on \( W^{1/2}H^2(\mathbb{C}^n) \) and equipped with the \( L^2 \)-norm has norm equal to 1.

4. \( \| H_{\Psi}W^{-1/2}|W^{1/2}H^2(\mathbb{C}^n)|_e \| < 1 \)

Hence, the space of maximizing vectors of \( H_{\Psi}W^{-1/2} \) is finite dimensional.
A taste of the proof of \( \dim \ker T_{\Phi - Q} < \infty \)

Let \( \Psi = \Phi - Q \) and \( W = \Psi^*\Psi \). Then

1. \( \ker T_\Psi = \{ f \in H^2(\mathbb{C}^n) : \| H_\Psi f \|_2 = \| \Psi \|_2 \} \)

2. \( W \) is invertible a.e. on \( \mathbb{T} \) and 
   \[ \| W(\zeta)^{-1} \| = s_{n-1}^{-1}(W(\zeta)) = t_{n-1}^{-2} \text{ for a.e. } \zeta \in \mathbb{T} \]

3. \( W^{1/2} \ker T_\Psi = W^{1/2} \{ f \in H^2(\mathbb{C}^n) : \| H_\Psi f \|_2 = \| \Psi f \|_2 \} \) 
   = \( W^{1/2} \{ f \in H^2(\mathbb{C}^n) : \| H_\Psi f \|_2 = \| W^{1/2} f \|_2 \} \) 
   = \{ \xi \in W^{1/2} H^2(\mathbb{C}^n) : \| H_\Psi W^{-1/2} \xi \|_2 = \| \xi \|_2 \} \)

**Conclusion:** The operator \( H_\Psi W^{-1/2} \) defined on \( W^{1/2} H^2(\mathbb{C}^n) \) and equipped with the \( L^2 \)-norm has norm equal to 1.

4. \( \| H_\Psi W^{-1/2} | W^{1/2} H^2(\mathbb{C}^n) \|_e < 1 \)

Hence, the space of maximizing vectors of \( H_\Psi W^{-1/2} \) is finite dimensional.
A taste of the proof of \( \dim \ker T_{\Phi - Q} < \infty \)

Let \( \Psi = \Phi - Q \) and \( W = \Psi^* \Psi \). Then

1. \( \ker T_{\Psi} = \{ f \in H^2(\mathbb{C}^n) : \| H_{\Psi} f \|_2 = \| \Psi \|_2 \} \)

2. \( W \) is invertible a.e. on \( \mathbb{T} \) and
   \[ \| W(\zeta)^{-1} \| = s_{n-1}^{-1}(W(\zeta)) = t_{n-1}^{-2} \text{ for a.e. } \zeta \in \mathbb{T} \]

3. \( W^{1/2} \ker T_{\Psi} = W^{1/2} \{ f \in H^2(\mathbb{C}^n) : H_{\Psi} f \|_2 = \| \Psi f \|_2 \} \)
   \[ = W^{1/2} \{ f \in H^2(\mathbb{C}^n) : H_{\Psi} f \|_2 = \| W^{1/2} f \|_2 \} \]
   \[ = \{ \xi \in W^{1/2} H^2(\mathbb{C}^n) : H_{\Psi} W^{-1/2} \xi \|_2 = \| \xi \|_2 \} \]

**Conclusion:** The operator \( H_{\Psi} W^{-1/2} \) defined on \( W^{1/2} H^2(\mathbb{C}^n) \) and equipped with the \( L^2 \)-norm has norm equal to 1.

4. \( \| H_{\Psi} W^{-1/2} \|_e < 1 \)

Hence, the space of maximizing vectors of \( H_{\Psi} W^{-1/2} \) is finite dimensional.
Can we compute the index of $T_{\Phi - Q}$?

**Question:** $\text{ind } T_{\Phi - Q} = 2k + \mu$?

Let $\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{z}^5 + \frac{1}{3} \bar{z} & -\frac{1}{3} \bar{z}^2 \\ \bar{z}^4 & \frac{1}{3} \bar{z} \end{pmatrix}$. Then

$$s_0(\mathcal{H}_\Phi) = \frac{\sqrt{10}}{3}, \quad s_1(\mathcal{H}_\Phi) = s_2(\mathcal{H}_\Phi) = s_3(\mathcal{H}_\Phi) = 1,$$

$$s_4(\mathcal{H}_\Phi) = \frac{1}{\sqrt{2}}, \quad \text{and } s_5(\mathcal{H}_\Phi) = \frac{1}{3},$$

and so $2k + \mu = 5$, where $\mu$ is the multiplicity of $s_1(\mathcal{H}_\Phi)$.

The superoptimal approximant of $\Phi$ with at most 1 pole is $Q = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{3} \bar{z} & 0 \\ 0 & 0 \end{pmatrix}$.

However, $\text{ind } T_{\Phi - Q} = \text{dim ker } T_{\Phi - Q} = 4$ even though $2k + \mu = 5$!
Can we compute the index of $T_{\Phi - Q}$?

**Question**: $\text{ind } T_{\Phi - Q} = 2k + \mu$?

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Alberto A. Condori  
Approximation by meromorphic matrix-valued functions
A special subspace

Let $B$ and $\Lambda$ be Blaschke-Potapov products such that
\[
\ker H_Q = BH^2(\mathbb{C}^n) \quad \text{and} \quad \ker H_{Q^t} = \Lambda H^2(\mathbb{C}^n).
\]

Let
\[
\mathcal{E} = \{ \xi \in \ker H_Q : \|H_{\Phi}\xi\|_2 = \|(\Phi - Q)\xi\|_2 \}
\]
and
\[
U = \Lambda^t(\Phi - Q)B.
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**Theorem (A.C. (2012))**

If the number of superoptimal singular values of $\Phi$ of degree $k$ equals $n$, then

1. $\mathcal{E} = B \ker T\!U$,
2. the Toeplitz operator $T\!U$ is Fredholm and
3. $\text{ind } T\!U = \dim \ker T\!U \geq n$. 
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**Theorem (A.C. (2012))**

Let \( \mathcal{E} = \{ \xi \in \ker H_Q : \| H_\Phi \xi \|_2 = \| (\Phi - Q) \xi \|_2 \} \). Then the Toeplitz operator \( T_{\Phi - Q} \) is Fredholm and has index

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\text{ind } T_{\Phi - Q} = 2k + \dim \mathcal{E}.
\]

In particular, \( \dim \ker T_{\Phi - Q} \geq 2k + n \).

**Corollary (A.C. (2012))**

If all superoptimal singular values of degree \( k \) of \( \Phi \) are equal, then

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holds, where \( \mu \) denotes the multiplicity of the singular value \( s_k(H_\Phi) \).
## The index formula

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Open problem #1

Sharp estimates on the “degree” of $Q$:

**Theorem (Peller-Vasyunin (2007))**

If $\Phi$ is a rational function $2 \times 2$ with poles off $\mathbb{T}$, then “generically” the best analytic approximant $Q$ to $\varphi$ is a rational function and

$$\deg Q \leq \deg \Phi - 2 \text{ unless } \Phi \in H^\infty(M_2).$$

In general, one has

$$\deg Q \leq 2 \deg \Phi - 3$$

and this inequality is sharp!

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Thank you!
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Thank you!