Approximation by meromorphic matrix-valued functions

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Our plan

1. Nehari-Takagi problem for scalar-valued functions
    1. Generalization of Classical Interpolation Problems
    2. Operator Theory
2. Matrix-valued functions: Are there suitable analogs of all previous results?

The case of scalar-valued functions

A class of meromorphic functions

Throughout,

1. L^∞ is the space of bounded functions on the unit circle $\mathbb{T}$;
2. H^∞ is the Hardy space of the unit disc $\mathbb{D}$ consisting of L^∞ functions whose Fourier coefficients of negative index vanish;
3. for $k \geq 0$, $\mathcal{B}_k$ is the set of Blaschke products of degree at most k, i.e. $b \in \mathcal{B}_k$ if and only if $b(\zeta) = c \prod_{j=1}^{d} \frac{\zeta - \lambda_j}{1 - \bar{\lambda}_j \zeta}$ where $d \leq k$, $\lambda_j \in \mathbb{D}$, and $c \in \mathbb{T}$; and
4. for $k \geq 0$, $H^\infty_{(k)} \stackrel{\text{def}}{=} \mathcal{B}_k^{-1} H^\infty$ is the set of meromorphic functions with at most k poles in the unit disc $\mathbb{D}$ (counting multiplicities) which are bounded near the unit circle $\mathbb{T}$.

The Nehari-Takagi problem

Let $\varphi \in L^\infty$. The *Nehari-Takagi problem* is to find a $q \in H^\infty_{(k)}$ which is closest to φ with respect to the L^∞-norm, i.e. to find $q \in H^\infty_{(k)}$ such that

$$\|\varphi - q\|_\infty = \mathrm{dist}_{L^\infty}(\varphi, H^\infty_{(k)}) = \inf_{f \in H^\infty_{(k)}} \|\varphi - f\|_\infty.$$

Any such function q is called a *best approximant in $H^\infty_{(k)}$ to φ*.

Question 1. *Existence?* Yes.

Question 2. *Uniqueness?* Not always, but a sufficient condition is continuity of φ on $\mathbb{T}$.

Who cares?

1. Mathematicians who enjoy operator theoretic complex function theory
   1. Generalization of Classical Interpolation Problems
   2. Connections with Operator Theory
2. Engineers interested in H^∞-control and signal theory
   1. systems can be described in the "frequency domain" as multiplication operator by a "transfer function" which belongs to certain Hardy casses if the system has certain stability properties
   2. rational functions are the "transfer functions" of systems having finite-dimensional state-space (and so these can be handled in practice)

Carathéodory-Fejér (CF) Interpolation

Given $c_0, c_1, \ldots, c_n \in \mathbb{C}$, the CF-problem is to find out when there is an $f \in H^\infty$ such that

$$\hat{f}(j) = c_j,\ 0 \leq j \leq n, \ \text{ and } \ \|f\|_\infty \leq 1.$$

Consider the function $g = c_0 + c_1 z + \ldots + c_n z_n$. Then any function whose Taylor coefficients are $c_0, c_1, \ldots, c_n$ is of the form $g + z^{n+1}h$ for some $h \in H^\infty$. Therefore, the CF-problem is solvable iff

$$1 \geq \inf\{\|g + z^{n+1}h\|_\infty : h \in H^\infty\} = \mathrm{dist}_{L^\infty}(\bar{z}^{n+1}g).$$

Nevanlinna-Pick (NP) Interpolation

Given distinct points $\lambda_i \in \mathbb{D}$, $1 \le i \le n$, and points $w_i \in \mathbb{C}$, $1 \le i \le n$, the NP-problem is to find out when there is an $f \in H^\infty$ such that

$$f(\lambda_i) = w_i,\ 1 \le i \le n, \quad \text{and} \quad \|f\|_\infty \le 1.$$

Consider the functions

$$B_i = \prod_{j \ne i} \frac{z - \lambda_j}{1 - \bar{\lambda}_j z}, 1 \le i \le n, \quad \text{and} \quad g = \sum_{i=1}^{n} \frac{w_i}{B_i(\lambda_i)} B_i.$$

Then any function that interpolates w_i at λ_i is of the form $g - Bh$ for some $h \in H^\infty$, where B a Blaschke product whose zeros are precisely $\lambda_1, \ldots, \lambda_n$. Thus, the NP-problem has a solution iff

$$1 \ge \inf\{\|g - Bh\| : h \in H^\infty\} = \operatorname{dist}_{L^\infty}(\bar{B}g, H^\infty).$$

Connections with Operator Theory: Hankel and Toeplitz

For $f \in L^2$, let $\mathbb{P}_+$ and $\mathbb{P}_-$ denote the orthogonal projections onto $H^2 = \{f \in L^2 : \hat{f}(n) = 0 \text{ for } n < 0\}$ and $H^2_- = L^2 \ominus H^2$, i.e.

$$\mathbb{P}_+ f = \sum_{n \geq 0} \hat{f}(n) z^n \quad \text{and} \quad \mathbb{P}_- f = \sum_{n < 0} \hat{f}(n) z^n.$$

Given $\varphi \in L^\infty$, we define the *Hankel operator* H_φ and the *Toeplitz operator* T_φ by

$$H_\varphi f = \mathbb{P}_- \varphi f \quad \text{and} \quad T_\varphi f = \mathbb{P}_+ \varphi f, \qquad f \in H^2,$$

respectively.

Some classical results

1. Kronecker (1881): A Hankel operator H_q has finite rank if and only if $q \in H^\infty_{(k)}$. Consequently, $\mathbb{P}_- q$ is a rational function whose poles lie in $\mathbb{D}$ and $\operatorname{rank} H_q = \deg \mathbb{P}_- q$.
2. Nehari (1957): $\|H_\varphi\| = \operatorname{dist}_{L^\infty}(\varphi, H^\infty)$
3. Hartman (1958): $\|H_\varphi\|_{\mathrm{e}} = \operatorname{dist}_{L^\infty}(\varphi, H^\infty + C)$, where $\|T\|_{\mathrm{e}}$ denotes the essential norm of the operator T, i.e.
$$\|T\|_{\mathrm{e}} \stackrel{\text{def}}{=} \inf\{\|T - K\| : K \text{ is compact}\}.$$
4. Adamyan, Arov, Krein (1970): *Existence* of a best approximants in $H^\infty_{(k)}$ and $s_k(H_\varphi) = \operatorname{dist}_{L^\infty}(\varphi, H^\infty_{(k)})$, where $s_k(T)$ denotes the kth singular value of the operator T, i.e.
$$s_k(T) \stackrel{\text{def}}{=} \inf\{\|T - R\| : \operatorname{rank} R \le k\}.$$

Characterization of the best meromorphic approximant

A function $\varphi \in L^\infty$ is called *k-admissible* if $\|H_\varphi\|_{\mathrm{e}} < s_k(H_\varphi)$ holds. E.g. functions in $C \setminus H^\infty_{(k)}$ are admissible.

Theorem (Peller (1990))

Let φ be k-admissible. Then q is the unique best meromorphic approximant in $H^\infty_{(k)}$ to φ iff

1. *$\varphi - q$ has constant modulus on $\mathbb{T}$, and*
2. *the Toeplitz operator $T_{\varphi-q}$ is Fredholm and has index*

$$\operatorname{ind} T_{\varphi-q} = 2k + \mu,$$

where μ denotes the multiplicity of the singular value $s_k(H_\varphi)$ of the Hankel operator H_φ.

Why is this caracterization useful?

Perturbation of Singular Values of Hankel Operators - 1

Theorem (Peller (1990))

Let φ be k-admissible and q be the best approx. in $H^\infty_{(k)}$ to φ. Suppose s is a singular value of H_φ of multiplicity $\mu \geq 2$ and

$$s = s_k(H_\varphi) = s_{k+1}(H_\varphi) = \ldots = s_{k+\mu-1}(H_\varphi) > s_{k+\mu}(H_\varphi).$$

Let $\lambda \in \mathbb{D}$, $b = \frac{z-\lambda}{1-\bar{\lambda}z}$, and $\psi = \bar{b}\varphi$. Then the following hold:

1. *If λ is a zero of q, then*

$$s = s_k(H_\psi) = s_{k+1}(H_\psi) = \ldots = s_{k+\mu}(H_\psi) > s_{k+\mu+1}(H_\psi).$$

2. *If λ is not a zero of q, then*

$$s = s_{k+1}(H_\psi) = s_{k+2}(H_\psi) = \ldots = s_{k+\mu-1}(H_\psi) > s_{k+\mu}(H_\psi).$$

Perturbation of Singular Values of Hankel Operators - 2

Theorem (Peller (1990))

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$$s = s_k(H_\varphi) = s_{k+1}(H_\varphi) = \ldots = s_{k+\mu-1}(H_\varphi) > s_{k+\mu}(H_\varphi).$$

Let $\lambda \in \mathbb{D}$, $b = \frac{z-\lambda}{1-\bar{\lambda}z}$, and $\psi = b\varphi$. Then the following hold:

1. *If λ is a pole of q, then*

$$s = s_{k-1}(H_\psi) = s_k(H_\psi) = \ldots = s_{k+\mu-1}(H_\psi) > s_{k+\mu}(H_\psi).$$

2. *If λ is not a pole of q, then*

$$s = s_{k+1}(H_\psi) = s_{k+2}(H_\psi) = \ldots = s_{k+\mu-2}(H_\psi) > s_{k+\mu-1}(H_\psi).$$

The degree of the best meromorphic approximant

Theorem (Peller-Khrushchëv (1982))

If φ is a rational function with poles outside $\mathbb{T}$ of degree, then the best meromorphic approximant q in $H^\infty_{(k)}$ to φ is also rational and

$$\deg q \leq \deg \varphi - 1 \quad \textit{unless } \varphi \in H^\infty_{(k)}.$$

The case of matrix-valued functions

Why should we do this?
In systems theory,

1. scalar-valued functions correspond to single input - single output systems
2. matrix-valued functions correspond to multiple input - multiple output systems

Notation

1. $\mathbb{M}_n$ denotes the space of $n \times n$ matrices equipped with the operator norm $\|\cdot\|_{\mathbb{M}_n}$.

2. $L^\infty(\mathbb{M}_n)$ is equipped with $\|\Phi\|_\infty = \operatorname{ess\,sup}_{\zeta\in\mathbb{T}} \|\Phi(\zeta)\|_{\mathbb{M}_n}$.

3. $H^\infty_{(k)}(\mathbb{M}_n)$ consists of matrix-valued functions Q with at most k poles in $\mathbb{D}$.

4. $Q \in L^\infty(\mathbb{M}_n)$ is said to have **at most** k **poles in** $\mathbb{D}$ if there is a Blaschke-Potapov product B of degree k such that $QB \in H^\infty(\mathbb{M}_n)$.

What is a Blaschke-Potapov product?

A finite **Blaschke-Potapov product** is an $n \times n$ matrix-valued function of the form $B = UB_1 \dots B_m$, where

$$B_i(z) = \frac{z - \lambda_i}{1 - \bar{\lambda}_i z} P_i + (I - P_i)$$

with $\lambda_i \in \mathbb{D}$ and orthogonal projection P_i on $\mathbb{C}^n$, $1 \leq i \leq m$, and U is a unitary matrix. We define $\deg B = \sum_{i=1}^{m} \operatorname{rank} P_i$.

Why count poles in this way?

In nicer language:

$$B(z) = U_0 \begin{pmatrix} \frac{z - a_1}{1 - \bar{a}_1 z} & \mathbb{O} \\ \mathbb{O} & I_{n-1} \end{pmatrix} U_1 \dots U_{k-1} \begin{pmatrix} \frac{z - a_k}{1 - \bar{a}_k z} & \mathbb{O} \\ \mathbb{O} & I_{n-1} \end{pmatrix} U_k,$$

where $a_1, \dots, a_k \in \mathbb{D}$ and $U_0, U_1, \dots, U_k$ are constant $n \times n$ unitary matrices.

Nehari-Takagi problem

Definition

Given $\Phi \in L^\infty(\mathbb{M}_n)$, we say that Q is a **best approximation in $H^\infty_{(k)}(\mathbb{M}_n)$ to Φ** if Q has at most k poles and

$$\|\Phi - Q\|_{L^\infty(\mathbb{M}_n)} = \mathrm{dist}_{L^\infty(\mathbb{M}_n)}(\Phi, H^\infty_{(k)}(\mathbb{M}_n)).$$

As before, given $\Phi \in L^\infty(\mathbb{M}_n)$, we define

1. the **Toeplitz operator** $T_\Phi : H^2(\mathbb{C}^n) \to H^2(\mathbb{C}^n)$ by

$$T_\Phi f = \mathbb{P}_+ \Phi f \quad \text{for } f \in H^2(\mathbb{C}^n),$$

   and

2. **Hankel operator** $H_\Phi : H^2(\mathbb{C}^n) \to H^2_-(\mathbb{C}^n)$ by

$$H_\Phi f = \mathbb{P}_- \Phi f \quad \text{for } f \in H^2(\mathbb{C}^n).$$

Results for matrix-valued function

1. (Known to specialists) A Hankel operator H_Q has finite rank if and only if $Q \in H^\infty_{(k)}$. Consequently, $\mathbb{P}_- q$ is a rational function whose poles lie in $\mathbb{D}$ and rank $H_q = \deg \mathbb{P}_- q$.
2. Page (1970): $\|H_\Phi\| = \mathrm{dist}_{L^\infty}(\Phi, H^\infty(\mathbb{M}_n))$
3. Ball-Helton (1983), Treil (1986): *Existence* of a best approximants in $H^\infty_{(k)}$ and $s_k(H_\varphi) = \mathrm{dist}_{L^\infty}(\varphi, H^\infty_{(k)})$, where $s_k(T)$ denotes the kth singular value of the operator T, i.e.
$$s_k(T) \stackrel{\mathrm{def}}{=} \inf\{\|T - R\| : \operatorname{rank} R \le k\}.$$
4. Treil (1986): $\|H_\varphi\|_{\mathrm{e}} = \mathrm{dist}_{L^\infty}(\varphi, H^\infty + C)$, where $\|T\|_{\mathrm{e}}$ denotes the essential norm of the operator T, i.e.
$$\|T\|_{\mathrm{e}} \stackrel{\mathrm{def}}{=} \inf\{\|T - K\| : K \text{ is compact}\}.$$

How about uniqueness of a best meromorphic approximant in $H^\infty_{(k)}(\mathbb{M}_n)$?

Superoptimal meromorphic approximation in $H^\infty_{(k)}(\mathbb{M}_n)$

Definition (Young)

Let $k \geq 0$ and $\Phi \in L^\infty(\mathbb{M}_n)$. We say that Q is a superoptimal meromorphic approximant of Φ in $H^\infty_{(k)}(\mathbb{M}_n)$ if Q has at most k poles in $\mathbb{D}$ and minimizes the essential suprema of singular values $s_j\left((\Phi - Q)(\zeta)\right)$, $j \geq 0$, with respect to the *lexicographic* ordering:

$$Q \text{ minimizes } \operatorname*{ess\,sup}_{\zeta\in\mathbb{T}} s_0(\Phi(\zeta) - Q(\zeta)) \text{ on } H^\infty_{(k)}(\mathbb{M}_n)$$

$$\text{then... minimize } \operatorname*{ess\,sup}_{\zeta\in\mathbb{T}} s_1(\Phi(\zeta) - Q(\zeta))$$

$$\text{then... minimize } \operatorname*{ess\,sup}_{\zeta\in\mathbb{T}} s_2(\Phi(\zeta) - Q(\zeta)) \dots \text{ and so on.}$$

For $j \geq 0$, the number $t_j^{(k)} \stackrel{\text{def}}{=} \operatorname{ess\,sup}_{\zeta\in\mathbb{T}} s_j(\Phi(\zeta) - Q(\zeta))$ is called the *jth superoptimal singular value of* Φ *of degree* k.

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For $j \geq 0$, the number $t_j^{(k)} \stackrel{\text{def}}{=} \operatorname{ess\,sup}_{\zeta\in\mathbb{T}} s_j(\Phi(\zeta) - Q(\zeta))$ is called the *jth superoptimal singular value of Φ of degree k.*

k-admissibility and uniqueness of a superoptimal approximant

We say that Φ is *k-admissible* if $\|H_\Phi\|_{\mathrm{e}}$ is less than the smallest non-zero superoptimal singular value of Φ of degree k and $s_k(H_\Phi) < s_{k-1}(H_\Phi)$.

Theorem (Peller-Young (1994, 1996), Treil (1995))

If Φ is k-admissible, then Φ has a unique superoptimal meromorphic approximant in $H^\infty_{(k)}(\mathbb{M}_n)$ and $s_j(\Phi(\zeta) - Q(\zeta)) = t_j^{(k)}$ for a.e. $\zeta \in \mathbb{T}$, $j \geq 0$.

The Toeplitz operator with symbol $\Phi - Q$

Theorem (A.C. 2012)

Suppose

1. Φ *is* k*-admissible and*
2. Φ *has* n *non-zero superoptimal singular values of degree* k.

Then the Toeplitz operator $T_{\Phi-Q}$ *is Fredholm and*

$$\operatorname{ind} T_{\Phi-Q} = \dim\ker T_{\Phi-Q} > 0.$$

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A taste of the proof of $\dim \ker T_{\Phi - Q} < \infty$

Let $\Psi = \Phi - Q$ and $W = \Psi^* \Psi$. Then

1. $\ker T_\Psi = \{ f \in H^2(\mathbb{C}^n) : \|H_\Psi f\|_2 = \|\Psi\|_2 \}$
2. W is invertible a.e. on $\mathbb{T}$ and $\|W(\zeta)^{-1}\| = s_{n-1}^{-1}(W(\zeta)) = t_{n-1}^{-2}$ for a.e. $\zeta \in \mathbb{T}$
3. $W^{1/2} \ker T_\Psi = W^{1/2}\{ f \in H^2(\mathbb{C}^n) : \|H_\Psi f\|_2 = \|\Psi f\|_2 \}$
   $= W^{1/2}\{ f \in H^2(\mathbb{C}^n) : \|H_\Psi f\|_2 = \|W^{1/2} f\|_2 \}$
   $= \{ \xi \in W^{1/2} H^2(\mathbb{C}^n) : \|H_\Psi W^{-1/2} \xi\|_2 = \|\xi\|_2 \}$

   Conclusion: The operator $H_\Psi W^{-1/2}$ defined on $W^{1/2} H^2(\mathbb{C}^n)$ and equipped with the L^2-norm has norm equal to 1.
4. $\|H_\Psi W^{-1/2} | W^{1/2} H^2(\mathbb{C}^n)\|_{\mathrm{e}} < 1$

Hence, the space of maximizing vectors of $H_\Psi W^{-1/2}$ is finite dimensional.

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Can we compute the index of $T_{\Phi-Q}$?

Question: ind $T_{\Phi-Q} = 2k + \mu$?

Let $\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{z}^5 + \frac{1}{3}\bar{z} & -\frac{1}{3}\bar{z}^2 \\ \bar{z}^4 & \frac{1}{3}\bar{z} \end{pmatrix}$. Then

$$s_0(H_\Phi) = \frac{\sqrt{10}}{3},\ s_1(H_\Phi) = s_2(H_\Phi) = s_3(H_\Phi) = 1,$$
$$s_4(H_\Phi) = \frac{1}{\sqrt{2}},\ \text{ and }\ s_5(H_\Phi) = \frac{1}{3},$$

and so $2k + \mu = 5$, where μ is the multiplicity of $s_1(H_\Phi)$.

The superoptimal approximant of Φ with at most 1 pole is $Q = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{3}\bar{z} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{pmatrix}$.

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A special subspace

Let B and Λ be Blaschke-Potapov products such that

$$\ker H_Q = BH^2(\mathbb{C}^n) \text{ and } \ker H_{Q^t} = \Lambda H^2(\mathbb{C}^n).$$

Let

$$\mathcal{E} = \{\xi \in \ker H_Q : \|H_\Phi \xi\|_2 = \|(\Phi - Q)\xi\|_2\}$$

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Theorem (A.C. (2012))

If the number of superoptimal singular values of Φ of degree k equals n, then

1. $\mathcal{E} = B \ker T_U$,
2. *the Toeplitz operator T_U is Fredholm and*
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Open problem #1

Sharp estimates on the "degree" of Q:

Theorem (Peller-Vasyunin (2007))

If Φ is a rational function 2×2 with poles off $\mathbb{T}$, then "generically" the best analytic approximant Q to φ is a rational function and

$$\deg Q \leq \deg \Phi - 2 \quad \textit{unless } \Phi \in H^{\infty}(\mathbb{M}_2).$$

In general, one has

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