Length, area, and closed curves in the plane

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Our plan

1. The problem and conjecture
2. Length and area
3. Rescaling of the problem
4. The special case
5. Questions and generalizations

1. The problem and conjecture

The problem

Question

Among all simple closed curves in the plane with a specified length ℓ, which one encloses the largest possible area?

So, after some thought, one obtains a natural candidate:

The circle.

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2. Length and area

Curves in $\mathbb{R}^2$

A *curve* $\vec{r}$ in $\mathbb{R}^2$ is a continuous (vector-valued) function $\vec{r} : [a, b] \to \mathbb{R}^2$.

So, $\vec{r}(t) = \langle x(t), y(t) \rangle$ for $a \leq t \leq b$, where x and y are continuous real-valued functions on $[a, b]$.

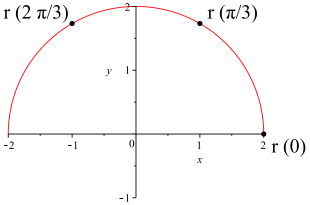

Figure: $\vec{r}(t) = \langle 2\cos(t), 2\sin(t) \rangle$ for $0 \leq t \leq \pi$

Image of a curve

Two *distinct* curves may trace out the same path in the plane.

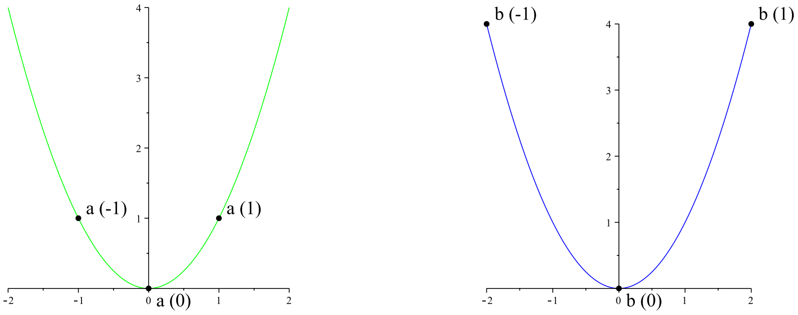

Figure: $\vec{a}(t) = \langle t, t^2 \rangle$ on $[0, 2]$ and $\vec{b}(t) = \langle 2t, 4t^2 \rangle$ on $[0, 1]$

Length of a curve

If the derivative $\vec{r}\,'$ exists and is continuous on $[a, b]$, then

$$\ell(\vec{r}) = \int_a^b |\vec{r}\,'(t)|\, dt.$$

Note: $|\vec{v}| = \sqrt{a^2 + b^2}$ when $\vec{v} = \langle a, b \rangle$.

Example

If $\vec{r}(t) = \langle 3\cos(t), 3\sin(t) \rangle$ *for* $0 \leq t \leq 2\pi$, *then*

$\vec{r}\,'(t) = \langle -3\sin(t), 3\cos(t) \rangle$, $\quad |\vec{r}\,'(t)| = 3 \quad$ *and so* $\quad \ell(\vec{r}) = 6\pi$.

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Re-parametrization by arc-length - Part 1

In particular, if $\vec{r}\,'$ is continuous and non-zero, then

1. $\varphi(t) = \int_a^t |\vec{r}\,'(\tau)|\, d\tau$,
2. $\varphi'(t) = |\vec{r}\,'(t)| > 0$,
3. $s = \varphi(t)$ is strictly increasing and so has an inverse function $t = \psi(s)$,and finally
4. the function $\vec{\gamma}(s) = (\vec{r} \circ \psi)(s)$ has derivative
$$\vec{\gamma}\,'(s) = \vec{r}\,'(\psi(s)) \cdot \psi'(s) = \vec{r}\,'(\psi(s)) \cdot \frac{1}{\varphi'(\psi(s))} = \frac{\vec{r}\,'(\psi(s))}{|\vec{r}\,'(\psi(s))|}.$$

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Re-parametrization by arc-length - Part 2

Conclusion: If $\vec{r}\,'$ is continuous and non-zero, then $\vec{r} : [a, b] \to \mathbb{R}^2$ admits a "re-parametrization" $\vec{\gamma} : [0, \ell] \to \mathbb{R}^2$ which has a unit tangent vector everywhere; that is,

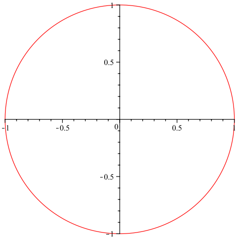
$$|\vec{\gamma}\,'(t)| = 1 \quad \text{for all } t \in (0, \ell),$$

where ℓ denotes the length of the curve.

The (new) curve $\vec{\gamma}(s)$ induced by $\vec{r}$ is called the *re-parametrization of $\vec{r}$ by arc-length*.

Simple closed curves

A *simple closed curve* is a curve $\vec{\gamma}$ in $\mathbb{R}^2$ that does not intersect itself and whose endpoints coincide.



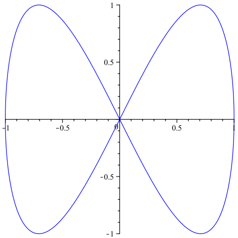


Figure: $\vec{a}(t) = \langle \cos(t), \sin(t) \rangle$ and $\vec{b}(t) = \langle \cos(t), \sin(2t) \rangle$ on $[0, 2\pi]$

Green's Theorem

Let $\vec{\gamma}$ be a simple closed curve with the counterclockwise orientation, and let D denote the bounded region enclosed by $\vec{\gamma}$. If $P(x, y)$ and $Q(x, y)$ are "smooth" functions, then

$$\oint_{\vec{\gamma}} (P(x,y)\,dx + Q(x,y)\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy.$$

A Consequence of Green's Theorem

$$\oint_{\vec{\gamma}} (P(x,y)\,dx + Q(x,y)\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy.$$

Choose $P = -y/2$ and $Q = x/2$. Then

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{1}{2} + \frac{1}{2} = 1$$

and so Green's Theorem becomes

$$\iint_D 1\,dx\,dy = \frac{1}{2} \oint_{\vec{\gamma}} (-y\,dx + x\,dy),$$

that is,

$$\text{Area}(D) = \frac{1}{2} \int_a^b (-y(t) \cdot x'(t) + x(t) \cdot y'(t))\,dt.$$

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So far...

1. Every (regular) curve $\vec{r} : [a, b] \to \mathbb{R}^2$ with length ℓ admits a "re-parametrization" $\vec{\gamma} : [0, \ell] \to \mathbb{R}^2$ such that

$$|\vec{\gamma}\,'(t)| = 1 \quad \text{for all } t \in (0, \ell).$$

2. If γ is a simple closed curve, then the bounded region D enclosed by γ equals

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3. Rescaling of the problem

The problem

Question

Among all simple closed curves in the plane with a specified length ℓ, which one encloses the largest possible area?

Conjecture:

$$\mathrm{Area}(D) \leq \frac{\ell^2}{4\pi}.$$

This is called the *isoperimetric inequality*.

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A simplification via re-scaling

For a curve $\vec{\gamma}(t) = \langle x(t), y(t) \rangle$ of length ℓ which enclosed a bounded region D, the isoperimetric inequality can be re-stated as

$$\frac{4\pi \,\mathrm{Area}(D)}{\ell^2} \leq 1.$$

It is important to observe that this quotient depends only on the "shape" of the curve γ and not on the "size" of the curve.

Consequence: It suffices to establish the inequality for simple closed curves of length 2π.

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4. The special case

A special case of the isoperimetric inequality

Theorem

Given a simple closed curve $\vec{\gamma} : [0, 2\pi] \to \mathbb{R}^2$ *such that*

$$|\vec{\gamma}\,'(t)| = 1 \qquad \textit{for all } t \in (0, 2\pi),$$

the bounded region D *enclosed by* γ *satisfies the inequality*

$$\mathrm{Area}(D) \leq \pi.$$

Moreover, equality holds if and only if γ *is a circle.*

From now on, $\vec{\gamma}$ is a curve that satisfies the hypothesis of the theorem. *How can we prove this?*

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"The shortest path between two truths in the real domain passes through the complex domain"

–J. Hadamard

Euler's formula comes into play...

An important example of a simple closed curve:

$\vec{\gamma}(t) = \langle \cos t, \sin t \rangle = \cos t + i \sin t = e^{it}$

Other important examples of simple closed curves:for each $n \in \mathbb{Z}$,

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Can any γ be generated by the examples above? More concretely, can we find coefficients a_n so that

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Some useful formulas

Let $g(t) = \sum_{m=-\infty}^{\infty} c_m e^{imt}$. Then the coefficients c_n can be computed from g:

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} g(t) e^{-int}\, dt.$$

Why are these representations useful?

The area of $\vec{\gamma}$ revisited

Recall that the area enclosed by $\vec{\gamma}(t) = \langle x(t), y(t) \rangle$ is given by

$$\text{Area}(D) = \frac{1}{2} \int_0^{\ell} (x(t) \cdot y'(t) - y(t) \cdot x'(t))\, dt.$$

Let

$$x(t) = \sum_{m=-\infty}^{\infty} a_m e^{imt} \quad \text{and} \quad y(t) = \sum_{m=-\infty}^{\infty} b_m e^{imt}.$$

Then

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Recall that $|\vec{\gamma}\,'(t)| = 1$ for all $t \in (0, 2\pi)$. So,

$$\frac{1}{2\pi}\int_0^{2\pi} |\vec{\gamma}\,'(t)|^2\,dt = 1$$

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The inequalities

$$2|\,\mathrm{Im}(a\overline{b})| \leq 2|a\overline{b}| = 2|a| \cdot |b| \leq |a|^2 + |b|^2$$

imply

$$\sum_{m=-\infty}^{\infty} m \cdot 2\,\mathrm{Im}(a_m \overline{b}_m) \leq \sum_{m=-\infty}^{\infty} |m| \cdot (|a_m|^2 + |b_m|^2)$$
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5. Questions and generalizations

Issues with our proof

Our proof raises some questions

1. How should one define "the region enclosed by a curve"?
2. What is area? Once one defines area appropriately, does this notion agree with the integral formula we used earlier?
3. Can our result be extended to more general types of curves? That is, can we remove the smoothness assumption?

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Generalized Isoperimetric Inequality

If Ω is any bounded domain in $\mathbb{R}^2$ whose boundary $\partial\Omega$ has finite length ℓ (and not necessarily a simple closed curve), it turns out that the following inequality holds:

$$\mathbf{m}_2(\Omega) \leq \frac{\ell^2}{4\pi}.$$

d-dimensional Isoperimetric Inequality

Theorem (Federer 1969)

For any set $\Omega \subseteq \mathbb{R}^d$ whose closure has finite Lebesgue measure, the following inequality holds

$$d\omega_d^{1/d}\, \mathbf{m}_d(\bar{\Omega})^{(d-1)/d} \leq \mathcal{M}_*(\partial\Omega),$$

where $\mathcal{M}_*(\partial\Omega)$ denotes the $d-1$ dimensional Minkowski content of $\partial\Omega$ and ω_d denotes the volume of the unit ball in $\mathbb{R}^d$.

An interesting consequence

The d-dimensional isoperimetric inequality is equivalent (for sufficiently smooth domains) to the *Sobolev inequality* on $\mathbb{R}^d$:

$$\left(\int_{\mathbb{R}^d} |u|^{\frac{d}{d-1}} \, d\, \mathbf{m}_d \right)^{\frac{d-1}{d}} \leq \frac{1}{d\omega_d^{1/d}} \int_{\mathbb{R}^d} |\nabla u| \, d\, \mathbf{m}_d$$

for all $u \in W^{1,1}(\mathbb{R}^d)$.

So, what have we learned today?

1. Translate a qualitative statement into a quantitative one
2. Green's Theorem is a powerful tool
3. "The shortest path between two truths in the real domain passes through the complex domain"
4. Translating a quantity from the continuous to the discrete via trigonometric series

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