Cyclicity in Dirichlet-type spaces and Optimal Polynomials

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This talk is based on the paper

Cyclicity in Dirichlet-type spaces and extremal polynomials,
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by C. Bénéteau, A.A. Condori, C. Liaw, D. Seco, and A.A. Sola.
(BCLSS).

The goals for this talk:

1. Describe what is done in the paper above (more or less)
2. Mention what results depend on the nature of the space D_α

Reproducing kernel Hilbert space

Let Ω be a set. We say $\mathcal{H}$ is a **reproducing kernel Hilbert space** (RKHS) on Ω if

1. $\mathcal{H}$ is a vector space consisting of functions $f : \Omega \to \mathbb{C}$,
2. $\mathcal{H}$ is a Hilbert space w.r.t. $\langle \cdot, \cdot \rangle$,
3. every point-evaluation functional (i.e. $\Phi_\lambda : \mathcal{H} \to \mathbb{C}$ is defined by $\Phi_\lambda(f) = f(\lambda)$ for $f \in \mathcal{H}$) is continuous.

In particular, for every $\lambda \in \Omega$, there is a unique vector $k_\lambda \in \mathcal{H}$ such that $f(\lambda) = \langle f, k_\lambda \rangle$ for all $f \in \mathcal{H}$.

Hilbert spaces of analytic functions

We say $\mathcal{H}$ is a **Hilbert space of analytic functions** on Ω if

1. $\mathcal{H}$ is a RKHS consisting of analytic functions on Ω,
2. $\mathcal{H}$ contains all analytic polynomials as a dense subset, and
3. $f \in \mathcal{H}$ implies $zf \in \mathcal{H}$.

The Weighted Hardy Space

Given $\beta = (\beta_n)_{n\geq 0}$ with $\beta_n > 0$, H^2_β consists of formal power series $f = \sum_{n\geq 0} a_n z^n$ such that $\|f\|^2_\beta = \sum_{n\geq 0} \beta_n^2 |a_n|^2 < \infty$. Thus,

1. H^2_β is a Hilbert space w.r.t. $\langle f, g\rangle = \sum_{n\geq 0} \beta_n^2 a_n \overline{b_n}$.
2. Each $f \in H^2_\beta$ has radius of convergence at least $R_\beta \stackrel{\text{def}}{=} \liminf_{n\to\infty} \beta_n^{-1/n}$, i.e. f is analytic on the disk of radius R_β.
3. H^2_β is a RKHS on $\Omega = \{\zeta : |\zeta| < R_\beta\}$; in fact, $k_\lambda = \sum_{n\geq 0} \frac{\overline{\lambda}^n}{\beta_n^2} z^n$ is the reproducing kernel at λ.

The Dirichlet-type spaces: $D_\alpha = H^2_\beta$, $\beta_n = \sqrt{(n+1)^\alpha}$

$D_{-1} = B =$Bergman, $D_0 = H^2 =$Hardy, and $D_1 = D$=Dirichlet.

Cyclicity

Let $\mathcal{H}$ be a Hilbert space of analytic functions on Ω. $f \in \mathcal{H}$ is called **cyclic** if $[f] = \mathcal{H}$, where

$$[f] \stackrel{\text{def}}{=} \overline{\text{span}}\{z^k f : k \geq 0\}.$$

Basic Observations:

1. The constant function $\mathbf{1}$ is cyclic.
2. $f \in \mathcal{H}$ cyclic implies

$$f(\zeta) \neq 0 \text{ for all } \zeta \in \Omega.$$

3. (Kopp 1969) D_α is an algebra when $\alpha > 1$. In particular, cyclic vectors are the invertible elements f in D_α, i.e. f has no zeros in the closed unit disk.
4. $f = 1 - z$ is cyclic in D even though it has a zero on $\mathbb{T}$.

Cyclicity and Approximation

(Brown-Shields 1984) f is cyclic in $\mathcal{H}$ if and only if $1 \in [f]$, i.e. $\exists (p_n)_{n\geq 0}$ of polynomials such that

$$\|1 - p_n f\| \to 0 \text{ as } n \to \infty. \tag{1}$$

This leads one[1] to ask:

1. If f is cyclic, can we produce $(p_n)_{n\geq 0}$ such that (1) holds?
2. Can we estimate the rate of decay of the norms in (1)?
3. What can we say about the approximating polynomials?

[1] Actually, five: C. Bénéteau, A.A. Condori, C. Liaw, D. Seco, and A.A. Sola

Optimal Approximants of Order n

We say that p_n^* is the **optimal approximant of order** n if

$$\|1 - p_n^* f\| = \text{dist}(1, f\mathcal{P}_n),$$

where $\mathcal{P}_n$ denotes the set of polynomials of degree at most n.

In particular, f is cyclic if and only if the sequence $(p_n^*)_{n\geq 1}$ satisfies $\|1 - p_n^* f\| \to 0$ as $n \to \infty$.

Estimates

When $f = 1 - z$ and $\mathcal{H} = D_\alpha$ with the "integral norm," BCLSS obtained

1. a formula for p_n^* (up to a constant factor),
2. $\operatorname{dist}^2_{D_\alpha}(1, (1-z)\mathcal{P}_n) \approx \dfrac{1}{(n+1)^{1-\alpha}}$ when $\alpha < 1$ and
3. $\operatorname{dist}^2_{D}(1, (1-z)\mathcal{P}_n) \approx \dfrac{1}{\log(n+1)}$ when $\alpha = 1$.

Question: Can one obtain *exact* formulas?

Theorem

If $\lambda \neq 0$ and $f = \lambda - z$, then

$$p_n^* = \sum_{\ell=0}^{n} \left(1 - \frac{H_\ell^{(\lambda)}}{H_{n+1}^{(\lambda)}}\right) \frac{1}{\lambda^{\ell+1}} z^\ell$$

and

$$\mathrm{dist}_{D_\alpha}(1, f\mathcal{P}_n) = \frac{1}{\sqrt{H_{n+1}^{(\lambda)}}},$$

where

$$H_\ell^{(\lambda)} = \sum_{k=0}^{\ell} \frac{|\lambda|^{2k}}{(k+1)^\alpha}.$$

Corollary

$f = 1 - z$ is cyclic in D_α precisely when $\alpha \leq 1$.

How about other Hilbert spaces of analytic functions?

Recall that if $f = 1 - z$ and $\mathcal{H} = D_\alpha$, we have

1. $\mathrm{dist}^2_{D_\alpha}(1, (1-z)\mathcal{P}_n) \approx \dfrac{1}{(n+1)^{1-\alpha}}$ when $\alpha < 1$ and
2. $\mathrm{dist}^2_{D}(1, (1-z)\mathcal{P}_n) \approx \dfrac{1}{\log(n+1)}$ when $\alpha = 1$.

Question: Can one get similar estimates for other functions?

Theorem (BCLSS)

Suppose f has zeros in $\mathbb{C} \setminus \mathbb{D}$ and at least one zero on $\mathbb{T}$. Then the same estimates hold if

- *f is a polynomial, or*
- *f admits analytic continuation to the closed disk.*

We asked:

1. If f is cyclic, can we produce $(p_n)_{n\geq 0}$ such that (1) holds?
2. Can we estimate the rate of decay of the norms in (1)?
3. What can we say about the approximating polynomials?

What about p_n^*?

In a Hilbert space of analytic functions $\mathcal{H}$ on $\mathbb{D}$, if f is cyclic, then

$$\lim_{n\to\infty} p_n^*(\zeta)f(\zeta) = 1 \text{ for } \zeta \in \mathbb{D}.$$

In particular, if f has a zero on $\mathbb{T}$, then $1/f$ has a power series with radius of convergence 1 and so (Jentzsch's theorem) every point of $\mathbb{T}$ is a limit point of the zeros of Taylor polynomials of $1/f$.

Question: Does the same occur for p_n^*? Can we find an asymptotic distribution of the zeros?

What about p_n^*?

Even if f is not cyclic, one can show that there is a function $f^* \in \mathcal{H}$ such that

$$\lim_{n\to\infty} p_n^*(\zeta)f(\zeta) = f^*(\zeta) \text{ for } \zeta \in \mathbb{D}.$$

For instance, if $\mathcal{H} = H^2$ and $f = \lambda - z$, then $[f] = b_\lambda H^2$ and

$$f^* = \bar{\lambda} b_\lambda \quad \text{where} \quad b_\lambda = \frac{\lambda - z}{1 - \bar{\lambda} z}.$$

Thus, it is possible that the polynomials p_n^* can be used to study invariant subspaces and factorizations in spaces of analytic functions.

Well, this requires further investigation!

Thank you!