Normal Matrices and Distance

Alberto A. Condori

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The Main Players

The collection of all $n \times n$ matrices $\mathbb{M}_n$ with complex entries:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix}$$

1. 0 and I denote the zero and identity matrices, respectively (sizes are understood in context).
2. What makes matrices interesting/disturbing?
    1. Multiplication is non-commutative in $\mathbb{M}_n$.
    2. Lack of cancellation law, e.g. $AB = CB$ does not imply $A = C$.
    3. There are (square) matrices without multiplicative inverses.
3. **Adjoint:** A^* denotes the conjugate transpose of $A \in \mathbb{M}_n$

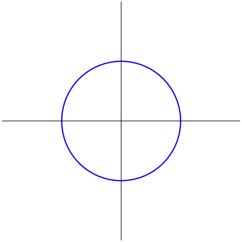
For vectors $v = (x_1, \ldots, x_n)$, we use the Euclidean Norm

$$|v| = \sqrt{|x_1|^2 + \ldots + |x_n|^2}.$$

For matrices, one often measures the "largest distortion" among unit vectors:

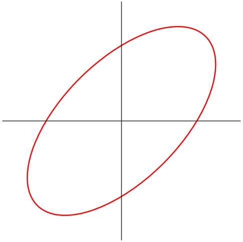
$$\|A\| = \max_{|v|=1} |Av|.$$

Exercise: $\displaystyle \min_{|v|=1} |Av| = \frac{1}{\|A^{-1}\|}$ for invertible matrices A.



$$\longrightarrow$$

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}$$



Spectrum

Let $T \in \mathbb{M}_n$. Assume there is a number λ and a non-zero vector v so that

$$Tv = \lambda v.$$

In this case, λ and v are called an **eigenvalue** and an **eigenvector** of T, respectively.

Importance. Linear Systems of DEs $y' = Ay$.

The set of all eigenvalues $\lambda_1, \ldots, \lambda_n$ of T is denoted by $\sigma(T)$ and is called the **spectrum** of T. Alternatively,

$$\begin{aligned} \sigma(T) &= \{\lambda : T - \lambda I \text{ is } \textbf{not} \text{ invertible}\} \\ &= \{\lambda : \det(T - \lambda I) = 0\}. \end{aligned}$$

Example: Compute $\|T\|$ and $\sigma(T)$ when T is a diagonal matrix.

Some Special Square Matrices

A matrix N is said to be **normal** if it commutes with its adjoint, i.e.

$$N^*N = NN^*.$$

A matrix N is said to be **unitary** if $U^*U = UU^* = I$.

Consequence: $\|UAV\| = \|A\|$ whenever U and V are unitary.

Spectral Theorem for Normal Matrices. T is normal if and only if there is a unitary matrix U such that $T = U\Lambda U^*$, where

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \quad \text{and} \quad \sigma(T) = \{\lambda_1, \dots, \lambda_n\}.$$

An Interesting Consequence and Question

Consequence: For normal matrices T,

$$\|(zI - T)^{-1}\| = \frac{1}{\mathrm{dist}(z, \sigma(T))} \quad \text{for all } z \notin \sigma(T). \qquad (1)$$

Thus, in a sense, *the spectrum completely captures the behavior of the norm of the resolvent* $\|(zI - T)^{-1}\|$ *for all* z.

> *Is the converse is true? That is, does equality in* (1) *imply that* T *is normal?*

There are 90 distinct characterizations of Normal matrices:

- R. Grone, C.R. Johnson, E.M. Sa, and H. Wolkowicz. **Normal Matrices.** Linear Algebra Appl. 87 (1987), 213-225.
- L. Elsner and Kh.D. Ikramov. **Normal matrices: an update.** Linear Algebra Appl. 285 (1998), no. 1-3, 291–303.

The following is based on joint work with C. Brooks (FGCU).

Theorem (C. Brooks & A.C.)

The following statements are equivalent for a matrix $T \in \mathbb{M}_n$.

1. *T is normal*
2. *For all $z \notin \sigma(T)$,*
$$\operatorname{dist}(z, \sigma(T)) = \|(zI - T)^{-1}\|^{-1}. \qquad (2)$$
3. *For each $1 \leq k \leq n-1$, there is a $z_k \in \mathbb{C}$ so that*
$$|z_k - \lambda_k| = \|(z_k I - T)^{-1}\|^{-1}.$$

Remark: For *any* $T \in \mathbb{M}_n$, $\operatorname{dist}(z, \sigma(T)) \geq \|(zI - T)^{-1}\|^{-1}$.

Corollary

If $T \in \mathbb{M}_2$ and $\operatorname{dist}(z, \sigma(T)) = \|(zI - T)^{-1}\|^{-1}$ holds for one point $z \notin \sigma(T)$, then T is normal.

What does the Corollary say about $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$?

Corollary

A matrix $T \in \mathbb{M}_n$ is normal if and only if

$$\sigma_{min}(zI - T) = \operatorname{dist}(z, \sigma(T)) \quad \textit{for all } z \notin \sigma(T).$$

Corollary

A matrix $T \in \mathbb{M}_n$ is normal if and only if

$$\|p(T)\| = \sup\{|p(\lambda)| : \lambda \in \sigma(T)\} \quad \textit{for every polynomial } p. \tag{3}$$

Pseudospectra & Norm Behavior

Given square matrices A and B (not necessarily of the same size) with **identical pseudospectra**[1], i.e.

$$\|(zI - A)^{-1}\| = \|(zI - B)^{-1}\| \quad \text{for all } z \in \mathbb{C}, \tag{4}$$

does it follow that A and B have the *same norm behavior*, that is,

$$\|p(A)\| = \|p(B)\|, \quad \text{for all polynomials } p? \tag{5}$$

Answer: *No.*

A. Greenbaum and L.N. Trefethen. **Do the pseudospectra of a matrix determine its behavior?** Technical Report TR 93-1371, Department of Computer Science, Cornell University, 1993.

[1] Convention: $\|(zI - A)^{-1}\| = \infty$ for $z \in \sigma(A)$.

Normality is respected by identical pseudospectra and same norm behavior

Corollary

Let A and B be square matrices[a]. Suppose that one of the following conditions holds:

1. *A and B have identical pseudospectra, or*
2. *A and B have the same norm behavior.*

If A is normal, then B is normal.

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[a] In this Corollary, the matrices A and B are *not* assumed to have the same size.

Seven conditions (necessary and sufficient) for unitary equivalence of a square matrix B to a normal matrix A were established by

T.G. Gerasimova. **Unitary similarity to a normal matrix.** Linear Algebra Appl. 436 (2012), no. 9, 3777–3783.

Corollary

Let $A, B \in \mathbb{M}_n$. The following statements are equivalent for a normal matrix A.

1. *A and B have the same norm behavior and characteristic polynomials.*
2. *A and B are unitarily equivalent.*

Open Problem

For $T \in \mathbb{M}_n$, define the function

$$\Phi_T(z) = \operatorname{dist}(z, \sigma(T)) \|(zI - T)^{-1}\| \quad \text{for } z \in \mathbb{C} \setminus \sigma(T).$$

An alternative reformulation of what we proved is:

If $\Phi_T(z) = 1$ for all $z \in \mathbb{C} \setminus \sigma(T)$, then T is normal.

Thus, we ask instead:

If $\Phi_T(z)$ is bounded on $z \in \mathbb{C} \setminus \sigma(T)$, must T be similar to a normal matrix?