Maximum principles for matrix-valued functions

Joint Mathematical Meetings - MAA General Session on Analysis

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Outline

Maximum Modulus Principle (MMP)

The MMP is key in Complex Analysis; e.g. it implies

1. the fundamental theorem of algebra,
2. the open mapping theorem,
3. Schwarz's lemma, etc.

Roughly, MMP states that analytic functions attain their maximum modulus on the boundary. More precisely:

Theorem (MMP)

If f non-constant and analytic on region Ω, then $|f(z)|$ cannot have a maximum in Ω.

This theorem is about *scalar-valued* functions: $f : \Omega \to \mathbb{C}$.
Throughout, Ω denotes a *region* of $\mathbb{C}$.

Applications require *matrix-valued* functions: $F : \Omega \to \mathbb{M}_n$

1. **Differential Equations**: growth/decay of the solutions $x(t) = \exp(tA)x_0$ to $\dot{x}(t) = A \cdot x(t)$ is controlled by the operator norm of the matrix-valued function $t \mapsto \exp(tA)$.
2. **Linear Algebra**: Given $A \in \mathbb{M}_n$, $\lambda \in \sigma(A)$ if and only if the *resolvent* $z \mapsto (A - zI)^{-1}$ has a singularity at λ, i.e. $A - \lambda I$ is not invertible. $\sigma(A)$ is *insufficient* for the analysis of a matrix, some focus on studying instead the operator norm of $z \mapsto (A - zI)^{-1}$, e.g. see [1].
3. **Mathematical Physics/Harmonic Analysis of Operators:** problems concerning spectral properties of an operator are often solved through the consideration of the "characteristic function."

Brooks, C. D., Condori, A. A. (2018). A resolvent criterion for normality. *Amer. Math. Monthly.* 125(2): 149–156.

Is there a *maximum-norm* principle?

Theorem (Maximum Norm Principle (see [1]))

Let $F : \Omega \to \mathbb{M}_n$ be analytic. If $\|F(z)\|$ attains its maximum in Ω, then $\|F(z)\|$ is constant on Ω.

Theorem (Maximum Frobenius-Norm Principle)

Let $F : \Omega \to \mathbb{M}_n$ be analytic. If $\|F(z)\|_{\mathcal{F}}$ assumes its maximum at some $z_0 \in \Omega$, then $F(z) = F(z_0)$ for all $z \in \Omega$.

Throughout, $\|A\| = \max\limits_{|v|=1} |Av|$ and $\|A\|_{\mathcal{F}} = \sqrt{\sum\limits_{i,j} |a_{i,j}|^2}$ denote the operator and Frobenius norms of $A \in \mathbb{M}_n$.

Dunford, N., Schwartz, J. T. (1958). *Linear Operators. I. General Theory.* New York–London: Interscience Publishers.

Example

Question: *What is an example of a non-constant matrix-valued function* $F : \Omega \to \mathbb{C}$ *so that* $\|F(z)\|$ *attains a maximum in* Ω*?*

The operator norm of the matrix-valued function $F : \mathbb{D} \to \mathbb{C}$ defined by

$$F(z) = \begin{bmatrix} 1 & 0 \\ 0 & g(z) \end{bmatrix} \tag{1}$$

satisfies

$$\|F(z)\| = \max\{1, |g(z)|\} = 1 \text{ for all } z \in \mathbb{D}$$

whenever $g : \mathbb{D} \to \mathbb{C}$ is analytic and $|g(z)| \leq 1$ for $z \in \mathbb{D}$ (e.g., $g(z) = z$). On the other hand, the Frobenius norm of $F(z)$ is non-constant:

$$\|F(z)\|_{\mathcal{F}} = \sqrt{1 + |g(z)|^2}.$$

Are there other examples?

Maximum Operator-Norm Principle: Our first version

Theorem

Let $F : \Omega \to \mathbb{M}_n$ *be analytic. If there is a* $z_0 \in \Omega$ *so that* $\|F(z)\| \leq \|F(z_0)\|$ *for all* $z \in \Omega$*, then there are* $n \times n$ *(constant) unitary*[a] *matrices* U *and* V*, and an analytic function* $G : \Omega \to \mathbb{M}_{n-1}$*, such that*

$$F(z) = U \begin{bmatrix} \|F(z_0)\| & 0 \\ 0 & G(z) \end{bmatrix} V. \tag{2}$$

[a] Recall that $A \in \mathbb{M}_n$ is said to be **unitary** if $A^*A = AA^* = I$.

Hence, the previously mentioned example is essentially the only example.

Consequence: Maximum Singular Values Principle

Recall that the singular values $s_k(A)$, $1 \leq k \leq n$, of an $n \times n$ matrix A are the non-negative square roots of the eigenvalues of A^*A ordered in the non-increasing order, that is,

$$s_1(A) \geq s_2(A) \geq \ldots \geq s_n(A).$$

In particular, $s_1(A) = \|A\|$ and $s_1^2(A) + s_2^2(A) + \cdots + s_n^2(A) = \|A\|_{\mathcal{F}}^2$.

Theorem

Let $F : \Omega \to \mathbb{M}_n$ be analytic. If $z \mapsto s_k(F(z))$ attains its maximum value on Ω for each $k = 1, \ldots, n$, then $F(z)$ is constant on Ω.

This might be considered a "perfect analog" of the MMP.

Maximum Frobenius-Norm Principle Revisited

Corollary

The following statements are equivalent for $F : \Omega \to \mathbb{M}_n$ analytic.

1. *For every $k = 1, \ldots, n$, $s_k(F(z))$ attains its max. at some $z_k \in \Omega$.*
2. *For every $k = 1, \ldots, n$, $s_k(F(z))$ is constant on Ω.*
3. *$\|F(z)\|_{\mathcal{F}}$ attains its maximum value at some $z_0 \in \Omega$.*
4. *$\|F(z)\|_{\mathcal{F}}$ is constant on Ω.*
5. *$F(z)$ is constant on Ω.*

Minimum Singular Values Principle

Theorem

If $F : \Omega \to \mathbb{M}_n$ is a non-constant analytic function, then no point $z_0 \in \Omega$ can be a minimum value for all of the functions $s_k(F(z))$, $1 \leq k \leq n$, unless $\det F(z_0) = 0$.

Alternatively, the Theorem states that if every function $s_k(F(z))$, $1 \leq k \leq n$, attains a minimum value at $z_0 \in \Omega$, then $s_n(F(z_0)) = 0$.

Example 1: $F(z) = \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix}$ over the unit disk

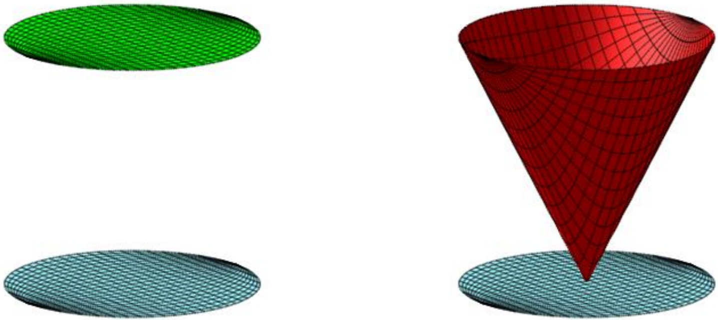

Figure: $s_1(F(z))$ is constant and $s_2(F(z))$ has a min at $z = 0$

Example 2: $F(z) = \begin{bmatrix} 1 & z \\ 0 & z-1 \end{bmatrix}$ over the unit disk

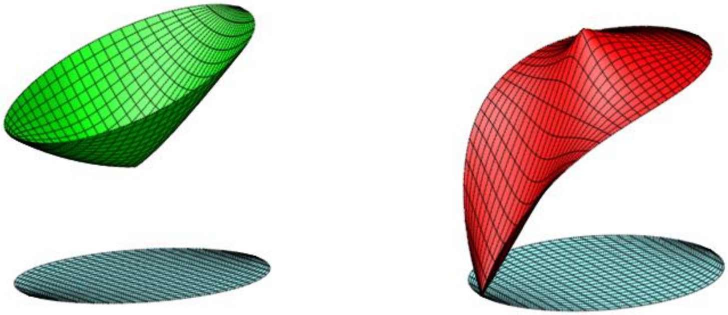

Figure: $s_1(F(z))$ has a min. at $z_1 = 0$ and $s_2(F(z))$ has a min. at $z_2 = 1$.

Return to resolvents

For $A \in \mathbb{M}_n$, $R_A(z) = (A - zI)^{-1}$ for $z \notin \sigma(A)$. Let

$$M_A = \sup_{\zeta \in \Omega} s_1(R_A(\zeta)) \quad \text{and} \quad m_A = \inf_{\zeta \in \Omega} s_n(R_A(\zeta)).$$

Theorem

If $A \in \mathbb{M}_n$ and Ω is any region of $\mathbb{C} \setminus \sigma(A)$, then

$$s_1(R_A(z)) < M_A \quad \textit{and} \quad s_n(R_A(\zeta)) > m_A \quad \textit{for all } z \in \Omega.$$

In particular, $s_1(R_A(z))$ and $s_n(R_A(z))$ are non-constant on any disk not containing any eigenvalue of A.

Return to matrix exponentials

For $A \in \mathbb{M}_n$, $\exp(zA) = \sum_{n=0}^{\infty} \frac{z^n}{n!} A^n$.

Since $\exp(zA)$ and $(A - zI)^{-1}$ are closely related, we ask:

1. Could $s_1(\exp(zA))$ and $s_n(\exp(zA))$ not attain their maximum or minimum values?
2. Could $s_1(\exp(zA))$ and $s_n(\exp(zA))$ be constant on a disk not containing any eigenvalue of A?

For answers, consider

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \exp(zA) = \begin{bmatrix} 1 & 0 \\ 0 & e^z \end{bmatrix}.$$

For more information, see my paper Maximum Principles for Matrix-Valued Analytic Functions to appear in **The American Mathematical Monthly**.

Thank you!