# Norm behavior of $3$-by-$3$ matrices

A. Condori and N. Seguin

Florida Gulf Coast University, Fort Myers, FL

## Applications and Differential Equations (DEs)

In various disciplines, the rates of change among various physical quantities are often interrelated and so modeled by systems of DEs.
Moreover, in applications, one is often interested in understanding the decay of the function(s) modeled.
For instance, the decay of the solution $x(t) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ to the $2$-by-$2$ system of DEs

$$\begin{cases} \dot{x}_1 = a_{11}x_1 + a_{12}x_2 \\ \dot{x}_2 = a_{21}x_1 + a_{22}x_2 \end{cases}$$

is well known to be bounded by the norm of

$$\exp(tA) = \sum_{k=0}^{\infty} \frac{t^k}{k!}A^k = I + tA + \frac{t^2}{2}A^2 + \frac{t^3}{6}A^3 + \dots,$$

where $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is the matrix of coefficients.

## Example (Trefethen 1992)

If $A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 5 \\ 0 & -2 \end{bmatrix}$, then

$$\exp(tA) = \begin{bmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{bmatrix}, \ \exp(tB) = \begin{bmatrix} e^{-t} & 5e^{-t}(1-e^{-t}) \\ 0 & e^{-2t} \end{bmatrix}.$$

Since the spectra

$$\sigma(A) = \{-1\} \quad \text{and} \quad \sigma(B) = \{-1, -2\},$$

*what might we predict about the behavior of the norms of the matrices* $\exp(tA)$ *and* $\exp(tB)$*?*

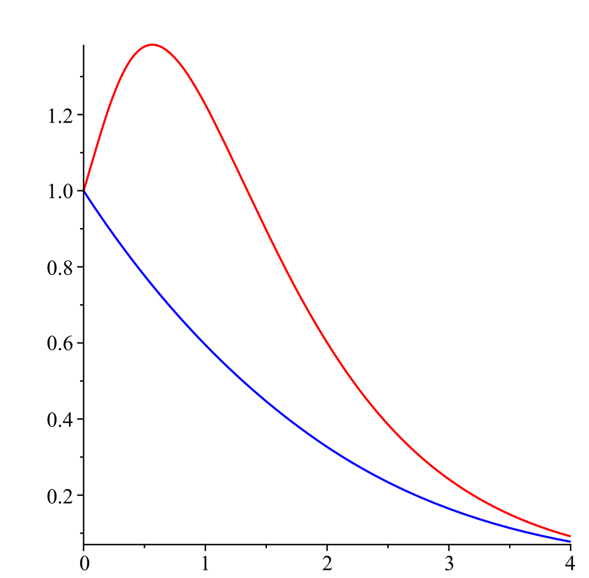


Figure 1: $||\exp(tA)||$ and $||\exp(tB)||$.
Which curve corresponds to $||\exp(tB)||$?

**Moral:** The spectrum alone is insufficient to understand "matrix behavior."

**Questions:** How can we predict when two matrices have the same behavior? What may be a suitable replacement for the spectrum?

## Norm behavior and pseudospectra

Let $A$ and $B$ be matrices. We say that $A$ and $B$ have the **same norm behavior** if

$$\|p(A)\| = \|p(B)\| \qquad (1)$$

for all polynomials $p(z) = a_0 + a_1z + a_2z^2 + \ldots + a_nz^n$.

***Problem:*** *Given two matrices, how can we determine whether they have the same norm behavior? That is, when does* (1) *hold?*

Since $\lambda \in \mathbb{C}$ satisfies $\|(A - \lambda I)^{-1}\| = \infty$ precisely when $\lambda$ is an eigenvalue of $A$, one may ask whether

$$\|(A - zI)^{-1}\| = \|(B - zI)^{-1}\| \ \text{ for all } z. \qquad (2)$$

When (2) holds, we say that $A$ and $B$ have **identical pseudospectra**.

## *Theorem (Our starting point)*

*The following statements are equivalent for* $2$*-by-*$2$ *matrices* $A$ *and* $B$*.*

1. $A$ *and* $B$ *have the same norm behavior.*
2. $A$ *and* $B$ *have identical pseudospectra.*
3. *The following three trace conditions hold:*

$$\operatorname{tr} A = \operatorname{tr} B, \ \operatorname{tr} A^2 = \operatorname{tr} B^2, \\ \operatorname{tr}(A^*A) = \operatorname{tr}(B^*B).$$

$1 \Leftrightarrow 2$ was stated by Greenbaum-Trefethen in the 90's. The version above was observed by Brooks-Condori a couple of years ago.

## Summer 2018 questions:

1. Must a pair of $3$-by-$3$ matrices have the same norm behavior precisely when they have identical pseudospectra?
2. Is it possible to find a simple "computational" criterion? How about a longer list of trace conditions?

To answer these questions, we constructed examples, counterexamples, and studied related work: Pearcy-Sibirskii seven trace conditions for unitary equivalence, Ransford's super-identical pseudospectra, etc. Then, we obtained

## *Theorem (Condori-Seguin)*

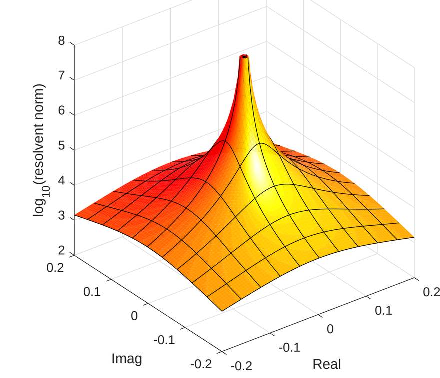
*For "derogatory"* $3$*-by-*$3$ *matrices* $A$ *and* $B$*, the following are equivalent.*

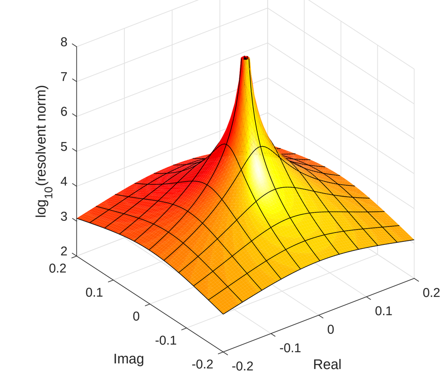
1. $A$ *and* $B$ *have the same norm behavior.*
2. $A$ *and* $B$ *have identical pseudospectra.*
3. $\|A - a \cdot I\|_F = \|B - b \cdot I\|_F$*, where* $a$ *and* $b$ *are the eigenvalues corresponding to* $A$ *and* $B$*, respectively, of largest multiplicity.*

The key was to find the "right example" and show that all other cases can be reduced to that example.

## Non-derogatory matrices

Could the Frobenius norm condition in the previous theorem be used in the non-derogatory case?





$$A = \begin{bmatrix} 0 & 8 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 7 & 4 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

This looks promising!
The matrices $A$ and $B$ satisfy the Frobenius-norm condition in the previous theorem, but they do not have the same norm behavior. Even worse, they do not even have identical pseudospectra!

***Problem:*** *Are there non-derogatory* $3$*-by-*$3$ *matrices with identical pseudospectra but without the same norm behavior?*

The next result "completes the picture" for $3$-by-$3$ matrices.

## *Theorem (Brooks-Condori)*

*For "non-derogatory"* $3$*-by-*$3$ *matrices* $A$ *and* $B$*, the following are equivalent.*

1. $A$ *and* $B$ *have the same norm behavior.*
2. $A$ *and* $B$ *have identical pseudospectra.*
3. *The following six trace conditions hold:*

$$\operatorname{tr} A = \operatorname{tr} B, \ \operatorname{tr} A^2 = \operatorname{tr} B^2, \ \operatorname{tr} A^3 = \operatorname{tr} B^3, \\ \operatorname{tr}(A^*A) = \operatorname{tr}(B^*B), \\ \operatorname{tr}(A^*A^2) = \operatorname{tr}(B^*B^2), \\ \operatorname{tr}(A^{*2}A^2) = \operatorname{tr}(B^{*2}B^2).$$

## Sanity Check!

***Question:*** *Can we use the six trace conditions in the derogatory case instead of the Frobenius conditions?*

*No.* For instance, the matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

are derogatory, have identical pseudospectra, and so the Frobenius norm condition holds; however, *none of the six trace conditions hold*. For instance, $\operatorname{tr} A = 1$ and $\operatorname{tr} B = 2$!

## Conclusions

Although matrices with identical spectra need not have the same norm behavior in general, we were successful in showing that matrices with identical pseudospectra do have the same norm behavior in the case of $3$-by-$3$ matrices. Moreover, we paraphrased this theoretical notion into easy-to-compute conditions that involve Frobenius norms or traces.

## Acknowledgements

This poster was based on joint work with C. Brooks (FGCU). We are grateful to the Seidler family and the Dean's Office in the College of Arts and Sciences for providing funding that facilitated this faculty-student research project during the summer of 2018 through the Seidler Fund.