Polynomially isometric matrices in low dimensions

Based on joint work with C. Brooks and N. Seguin

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Outline

1. Functional calculus of a matrix A and its importance
2. Notions of size/norm of a matrix A
3. Polynomially isometric matrices
4. Pseudospectra
5. Partial bad news
6. Partial good news
7. We end with *better news!*

Functional calculus of a matrix

Given an $d \times d$ matrix A, it is of importance to consider matrices that can be generated by A, e.g.

$$A^2, \quad A^3 + 4A^2, \quad I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \frac{1}{24}A^4 + \dots, \quad \text{etc.}$$

1. How can all such matrices be written concisely?
2. Why is this important?

First consumer: Linear Algebra

1. *Eigenvalues*: Every matrix A has an eigenvalue λ, i.e. there is a vector $v \neq 0$ so that $Av = \lambda v$.

   One-line proof: Given $w \neq 0$, there is a polynomial p so that $p(A)w = 0$.

2. *Numerical methods:* The convergence of algorithms to compute solutions to $Ax = b$ rely on estimates on the size of certain $p(A)$'s and on their rate of decay, e.g. GMRES,

$$\|Ax_m - b\| = \min\{\|p(A)x_0\| : \deg p \leq m, p(0) = 1\}.$$

Second consumer: Differential Equations

1. *Computation of solutions*: How to solve the DE below?

   $$y''(t) + c_1 y'(t) + c_0 y(t) = f(t)$$

   Easy: If $p(z) = z^2 + c_1 z + c_0 = (z - \lambda_1)(z - \lambda_2)$ and let D denotes the differentiation operator, the DE above states $p(D)y = f$.

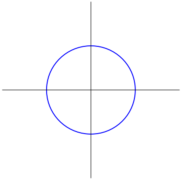
2. *Decay estimates:* Solutions to the system of DEs $x' = Ax$ depend on $\|f(A)\|$, where $f(t) = \exp(tA)$.

What is the size of a matrix?

For a vector $x = (x_1, x_2, \ldots, x_n)$, we use the *Euclidean norm*

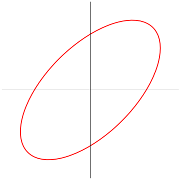
$$|x| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}.$$

Likewise, one may use the **Frobenius/Euclidean norm** $\|A\|_F$ which is easy to compute, but not useful in practice.
In practice, one needs to measure the "largest distortion" among unit vectors. The **spectral norm** $\|A\|$ of A is defined as $\|A\| = \max_{|v|=1} |Av|$.



$$\longrightarrow$$

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}$$



Exercises: (1) Show that $\|A\|_F = \sqrt{10}$ but $\|A\| = 2\sqrt{2}$.
(2) Show that the "smallest distortion" is $\min\limits_{|v|=1} |Av| = \dfrac{1}{\|A^{-1}\|}$.

Our main question

Given a pair of square matrices A and B, what set of invariants (e.g. eigenvalues, Frobenius norms, etc.) are necessary and sufficient to ensure that A and B are "polynomially isometric"?

We say that A and B are **polynomially isometric** if

$$\|p(A)\| = \|p(B)\| \quad \text{for } \textit{all} \text{ polynomials } p.$$

What quantity may capture the "spirit" of a matrix?

The spectrum of A

The **spectrum** $\sigma(A)$ of A is the set of its eigenvalues.

If you are familiar with the Spectral Theorem: *What do you think?*

Moral: Spectral analysis is ineffective

The matrices $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ have equal spectra, but $\|A\| = 1$ and $\|B\| = \sqrt{2}$, i.e. A and B are not polynomially isometric.

More generally, it is now well-known to numerical analysts that the spectrum is rarely sufficient to analyze a matrix. If so, what is a more effective replacement?

Pseudospectra

Roughly, a **pseudospectral plot** for a matrix A consists of contour plots of the norm $\|(zI - A)^{-1}\|$ of its resolvent $z \mapsto (zI - A)^{-1}$. A recent book by Trefethen and Embree illustrates that pseudospectra may capture the "spirit" of a (non-normal) matrix more effectively.

As before, $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$. Are their graphs *identical*?

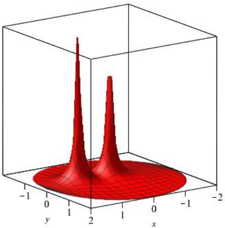


Figure: Sketch of $\|(zI - A)^{-1}\|$

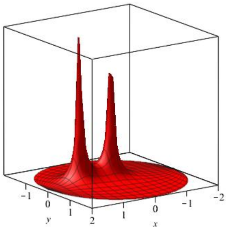


Figure: Sketch of $\|(zI - B)^{-1}\|$

Main Question

For A and B to be polynomially isometric, is it necessary and sufficient that A and B have identical pseudospectra?

Recall that A and B are **polynomially isometric** if

$$\|p(A)\| = \|p(B)\| \quad \text{for } \textit{all} \text{ polynomials } p. \tag{1}$$

Also, A and B have **identical pseudospectra** if

$$\|(zI - A)^{-1}\| = \|(zI - B)^{-1}\| \quad \text{for all } z \in \mathbb{C}. \tag{2}$$

The implication $1 \implies 2$ is due to Greenbaum & Trefethen (1993).

Moreover, $1 \iff 2$ holds if at least one of the matrices is normal; this is a consequence of previous work of Brooks and Condori (2018).

Negative results

1. An example due to Greenbaum and Trefethen (1993) shows that there are 5×5 matrices having identical pseudospectra for which the condition in (1) fails with $p(z) = z$.
2. An example due to Fortier Bourque and Ransford (2009) shows that there are 4×4 matrices having identical pseudospectra but whose squares have distinct norms, i.e. (1) fails with $p(z) = z^2$.

What about matrices of *lower* dimensions?

The case of 2×2 matrices

Theorem

The following statements are equivalent for $A, B \in \mathbb{M}_2$.

1. *A and B have identical pseudospectra.*
2. *A and B are polynomially isometric.*
3. *A and B are unitarily similar.*

"$1 \iff 2$" was observed by Greenbaum & Trefethen (1993).

A former FGCU student, N. Camacho and I also worked out a proof for his Senior Seminar project (2016); however, we did not realize the equivalence to 3 at time.

Translation to an "easy-to-check" criterion

Corollary

For $A, B \in \mathbb{M}_2$ to be polynomially isometric, it is necessary and sufficient that

$$\operatorname{tr} A^*A = \operatorname{tr} B^*B, \ \operatorname{tr} A = \operatorname{tr} B, \quad \textit{and} \quad \operatorname{tr} A^2 = \operatorname{tr} B^2.$$

Exercise 1: Why are $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ <u>not</u> polynomially isometric?

Exercise 2: Construct matrices $A, B \in \mathbb{M}_2$ which are similar but do not have identical pseudospectra.

The case of 3×3 matrices - Part 1

Theorem

The following statements are equivalent for $A, B \in \mathbb{M}_3$.

1. *A and B have identical pseudospectra and characteristic polynomials.*
2. $\operatorname{tr}(A^*A) = \operatorname{tr}(B^*B)$, $\quad \operatorname{tr}\left(A^*A^2\right) = \operatorname{tr}\left(B^*B^2\right)$,
   $\operatorname{tr}\left(A^{*2}A^2\right) = \operatorname{tr}\left(B^{*2}B^2\right)$, $\quad \operatorname{tr} A^k = \operatorname{tr} B^k$ *for* $k = 1, 2, 3$.

In particular, A and B must be polynomially isometric.

Exercise: Consider the matrices

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}. \tag{3}$$

1. Show that the matrices in (3) are polynomially isometric.
2. Are they unitarily similar?

The case of 3×3 matrices - Part 2

Question

In the previous theorem, is the requirement that A and B have the same characteristic polynomials really important?

To address this question, consider the matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Where does this leave us in our quest to characterize polynomially isometric matrices? Is having identical pseudospectra good enough?

The case of 3×3 matrices - Part 3

Theorem

The following statements are equivalent for $A, B \in \mathbb{M}_3$.

1. *A and B have identical pseudospectra.*
2. *A and B are polynomially isometric.*

Moreover, if A and B also have the same minimal polynomial of degree 2, then the above statements are equivalent to

3. $\|A - \gamma_A I\|_F = \|B - \gamma_B I\|_F$, *where γ_A and γ_B are the eigenvalues corresponding to A and B, respectively, of largest multiplicity.*

Recap

1. Functional calculus of a matrix A and its importance
2. Notions of size/norm of a matrix A
3. Polynomially isometric matrices
4. Pseudospectra (as a more effective replacement to spectra)
5. Partial bad news (counterxamples in dimensions 4 & 5)
6. Partial good news (success in the 2×2 case)
7. We now have *better news:* success in the 3×3 case with easy-to-check criteria!

Challenge: Test your knowledge!

Only two of the following seven matrices are polynomially isometric but *not* unitarily similar. Find them!

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad A_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_6 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$A_7 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

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