An improved first order local regularization method for ill-posed Volterra equations

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Abstract

In this paper, we present a first order local regularization method for solving ill-posed Volterra equations with $\nu$-smoothing kernels, and establish stability and convergence of the method for all values of $\nu \in \mathbb{N}$. The method is an improvement of one whose numerical performance is shown to erode at $\nu = 4$ and whose convergence theory is uncertain once $\nu > 4$. We describe numerical implementation of the fast sequential algorithm associated with the method and provide a new scheme to approximate the initial condition. Numerical examples illustrate our theoretical results, particularly the method’s stability in the case $\nu = 4$ and higher.

Keywords: ill-posed Volterra equation, inverse problem, local regularization, fast sequential algorithm

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1. Introduction

The purpose of this paper is to present a convergent first order local regularization method for solving the inverse problem

$Au = f$  \hspace{1cm} (1)

for $u$, given the data $f$ and the Volterra convolution operator $A$ acting on a Banach space of functions on an interval $I$ (specified below) containing the origin, defined by

$Au(t) := \int_0^t k(t - s)u(s) \, ds, \quad \text{a.e. } t \in I,$  \hspace{1cm} (2)

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with $\nu$-smoothing kernel $k$ for $\nu \in \mathbb{N}$; that is, $k \in C^{\nu}(I)$ and

$$
\begin{cases}
  k(0) \neq 0, & \text{if } \nu = 1, \\
  k^{(\ell)}(0) = 0, & \ell = 0, 1, ..., \nu - 2, \quad \text{and} \quad k^{(\nu-1)}(0) \neq 0, & \text{if } \nu > 1.
\end{cases}
$$

Without loss of generality, we assume $k^{(\nu-1)}(0) = 1$.

The localized approach herein involves the utilization of data on a future interval. Thus we assume for simplicity that equation (1) holds on an extended, fixed interval $[0, 1 + \bar{\alpha}]$, for some $\bar{\alpha} > 0$, and its solution is sought on the interval $[0, 1]$. Denote the intervals

$I := [0, 1 + \bar{\alpha}], \quad I_0 := [0, 1], \quad \Lambda := (0, \bar{\alpha}]$.

Henceforth, let $p \in (1, \infty)$ and note that $\mathcal{A} \in \mathcal{L}(L^p(I))$ is a bounded linear operator on $L^p(I)$. We observe that $\mathcal{A}$ is compact [4, Chapter 2, Theorem 2.5] and injective (see the differentiation arguments in [4] and [6]), from which we establish that $\mathcal{A}^{-1}$ is unbounded on $L^p(I)$, and furthermore that the inverse problem (1) is ill-posed ([11, Sec. 3.1.1]). (More about $\nu$-smoothing kernels and their implications in solving (1) can be found in the survey paper [6].)

In applications, the data $f \in \mathcal{R}(\mathcal{A}) \subset L^p(I)$ is rarely known exactly and so, given $\delta > 0$, we must instead make use of perturbed, or measured data $f^\delta \in L^p(I)$ which we assume satisfies

$$
\|f^\delta - f\|_{L^p(I)} \leq \delta.
$$

We briefly take up the case in which the problem is set in the space $C(I)$ in Section 4 below in order to draw comparisons between our results and those in [9]. Regardless of which topology we use, the presence of perturbed data means that a regularization method must be employed. Such schemes produce stable, (regularization)-parameter dependent, and reasonably accurate approximates to the solution of (1) using the inexact data $f^\delta$.

Our proposed method is both an extension and improvement of the so-called “first order sequential predictor–corrector” method by Ring in [9] for which the main shortcoming was a lack of stable convergence in the cases of $\nu > 4$. In recent years, the term “local” has replaced “sequential predictor–corrector” to emphasize the manner in which data is used and approximate solutions are constructed. Such methods offer improvements in algorithmic speed over classical regularization methods such as those developed by Tikhonov and Lavrentiev.

Indeed, classical methods are generally inefficient when applied to Volterra inverse problems; for example, Tikhonov regularization of equation (1) makes use of data on the entire interval $[0, 1]$ to approximate the solution $u$ at a single point $t$ in that interval (see e.g. [1]). Volterra equations are causal in nature so only the data on $[t, 1]$ is needed to estimate $u(t)$ and, in fact, excellent results can be obtained using data restricted to an interval $[t, t + \alpha]$ for some small $\alpha > 0$. It is this use of localized data and an affine approximation on the same interval that leads to the first order local regularization equation under consideration in
this paper. As seen in [9] and Section 5, this approach results in a sequential algorithm that is faster than those associated with classical methods.

The first step in the derivation of a locally regularized equation is to define an operator used for “sampling” or “filtering” the data on a future interval of length \( \alpha \in \Lambda \). Thus we define \( T_{\alpha} : L^p(I) \rightarrow L^p(I_0) \) by

\[
T_{\alpha} f(t) := \int_0^\alpha f(t + \rho) \, d\eta_{\alpha}(\rho), \quad \text{a.e.} \ t \in I_0,
\]

where \( \eta_{\alpha} \) is a suitably defined measure (cf. Section 2). We apply \( T_{\alpha} \) to both sides of (1), yielding

\[
\int_0^\alpha \int_0^{t+\rho} k(t + \rho - s)u(s) \, ds \, d\eta_{\alpha}(\rho) = T_{\alpha} f(t), \quad \text{a.e.} \ t \in I_0,
\]

which may be rewritten

\[
D_{\alpha} u + A_{\alpha} r u = T_{\alpha} f,
\]

where the operators \( D_{\alpha}, r : L^p(I) \rightarrow L^p(I_0) \) are defined by

\[
D_{\alpha} u(t) := \int_0^\alpha \int_0^\rho k(\rho - s)u(t + s) \, ds \, d\eta_{\alpha}(\rho), \quad \text{a.e.} \ t \in I_0,
\]

\[
r u := u|_{I_0}, \ u \in L^p(I),
\]

and \( A_{\alpha} : L^p(I_0) \rightarrow L^p(I_0) \) is a new Volterra operator given for \( u \in L^p(I_0) \) by

\[
A_{\alpha} u(t) := \int_0^t k_{\alpha}(t - s)u(s) \, ds, \quad k_{\alpha}(t) := \int_0^\alpha k(t + \rho) \, d\eta_{\alpha}(\rho), \quad \text{a.e.} \ t \in I_0.
\]

An \( m \)th order local regularization equation results from (4) with the replacement of \( f \) by \( f^\delta \) and the use of an \( m \)th order Taylor expansion of \( u(t + s) \) in the integrand of \( D_{\alpha} \). This leads to the regularized equation

\[
\sum_{\ell=0}^m u^{(\ell)}(t) \left[ \frac{1}{\ell!} \int_0^\alpha \int_0^\rho k(\rho - s)s^{\ell} \, ds \, d\eta_{\alpha}(\rho) \right] + A_{\alpha} u(t) = T_{\alpha} f^\delta(t), \quad \text{a.e.} \ t \in I_0,
\]

for \( u \) defined on \( I_0 \), which for \( m \geq 1 \) also requires suitable initial condition(s) on \( u \). In the case \( m = 0 \), the zeroth order local regularization equation is a second kind Volterra integral equation,

\[
a_{\alpha} u(t) + A_{\alpha} u(t) = T_{\alpha} f^\delta(t), \quad \text{a.e.} \ t \in I_0,
\]

which was studied in [7] for continuous data and in [1, 2] for \( L^p \) data (with discrepancy principle in the latter). In the case \( m = 1 \), we obtain the first order method from [9] that is the subject of this paper, namely, the Volterra integrodifferential initial value problem

\[
a_{\alpha} u(t) + b_{\alpha} u'(t) + A_{\alpha} u(t) = T_{\alpha} f^\delta(t), \quad \text{a.e.} \ t \in I_0,
\]

\[
u(0) = u_0,
\]
for $u_0$ an appropriate approximation of the true solution at $t = 0$. Here the scalars $a_\alpha$ in (8), (9) and $b_\alpha$ in (9) are given by

$$a_\alpha := \int_0^\alpha \int_0^\rho k(\rho - s) \, ds \, d\eta_\alpha(\rho),$$

(11)

$$b_\alpha := \int_0^\alpha \lambda(\rho) \, d\eta_\alpha(\rho), \quad \lambda(\rho) := \int_0^\rho (\rho - s) k(s) \, ds.$$  

(12)

One possible approximation for $u_0 = u_0(\alpha, \delta)$ is motivated in [9] by a numerical algorithm for the solution of the first order regularization equation (9), leading to the 2 × 2 system

$$\begin{pmatrix} c_\alpha & a_\alpha \\ a_\alpha & b_\alpha \end{pmatrix} \begin{pmatrix} u(0) \\ u'(0) \end{pmatrix} = \begin{pmatrix} \tilde{g}_\alpha^\delta \\ T_\alpha f^\delta(0) \end{pmatrix},$$

(13)

where the first component of the vector solution defines $u_0$ (the second component is not used), and $c_\alpha$, $\tilde{g}_\alpha^\delta$ are defined below in (28), (29), respectively.

We suggest as an alternative the simpler approach of taking the approximate initial value $u_0$ to be the solution at $t = 0$ of the zeroth order local regularization equation (8), leading to

$$u_0 = \frac{1}{a_\alpha} T_\alpha f^\delta(0).$$

(14)

Both approximations are studied in detail in Section 3.1.

Note that the general $m^{th}$ order local regularization equation (7) can be expressed, in the case of appropriate initial conditions, as

$$(a_\alpha L_\alpha + A_\alpha) u = T_\alpha f^\delta,$$

(15)

for $L_\alpha$ a densely defined linear operator. In the zeroth order equation, $L_\alpha$ is the identity operator. For first order regularization, we treat a nonzero initial value separately after a translation to zero initial value (cf. Section 3); in this case, $L_\alpha : \text{dom}(L_\alpha) \subseteq L^p(I_0) \mapsto L^p(I_0)$ is defined via

$$L_\alpha u := u + \frac{b_\alpha}{a_\alpha} u', \quad u \in \text{dom}(L_\alpha) := \{ u \in W^{1,p}(I_0), \, u(0) = 0 \}.$$  

(16)

The defining equations for the first order local regularization method are equations (9)–(10). A convergence theory for the method relies on a suitable construction of the measure $\eta_\alpha$ appearing in the definition of the sampling operator $T_\alpha$, one which depends on the underlying operator $\mathcal{A}$ and, in particular, on $\nu$ and the $\nu$-smoothing kernel $k$.

In [9], positive such measures were constructed, similar to those in [5] for zeroth order local regularization. The construction in [5] was shown by Ring and Prix in [10] to be implicated in numerical examples with unstable (and non-converging) reconstructions in the case of $\nu = 4$ and higher. In fact, they provide justification that no positive measure exists for which the sufficient
conditions of the zeroth order local method are satisfied when \( \nu > 4 \). Although reconstructions in [9] are excellent in numerical examples with one-, two-, and three-smoothing kernels, Ring’s first order method apparently suffers from these same limitations as illustrated therein by a numerical example with \( \nu = 4 \).

In this paper, we take an approach similar to Lamm’s in [7] in which a particular class of signed measures is used to overcome the limitations of the method in [5]. Such measures are easily constructed, and the resulting theory establishes stability and convergence of the first order method for all values of \( \nu \in \mathbb{N} \). In addition, as measured data is rarely smooth, we improve upon the requirement in [9] that \( f, f^3 \) be continuous by extending to the problem setting to \( L^p \).

1.1. Outline of the paper

In Section 2, we define the measure \( \eta_\alpha \) in terms of a family of signed measures \( \{w_\alpha\}_{\alpha \in \Lambda} \) that satisfy prescribed conditions, a combination of those appearing in [1, 7, 9]. In contrast to the positive measures used in [9], we argue that, for every \( \nu \in \mathbb{N} \), such a class of signed measures always exists and can be constructed in a stable manner. We then describe properties of the operators and provide estimates on scalars associated with the first order method that rely on these measures. We also verify two lemmas that are used to prove the main convergence theorem.

In Section 3, we show that the conditions prescribed on the measure are sufficient to establish stability and convergence of solutions of the first order local regularization equations (9)–(10) (derived using the new signed measure). We also prove convergence using each of the approximate initial values \( u_0 \) defined in (13) and (14) above and obtain an a priori convergence rate under the assumption of additional regularity on the exact solution.

In Section 4, we show that in the special case of continuous data \( f, f^3 \), our results are an improvement over those in [9].

In Section 5, we implement the method numerically using the same discretization and collocation employed previously in [9, Section 5] on equations with three-, four-, and five-smoothing kernels. The examples illustrate a drastic improvement in performance, particularly when the signed measure is constructed in a stable way. We demonstrate the convergence and stability of our method and the rate of convergence, and observe a slight visual effect of the alternative choice of approximate initial condition.

Finally, we mention how an \( m^{th} \) order local regularization theory can be established from the framework of the method herein. To that end, we provide sufficient conditions for convergence and a rate of convergence in a general context in Section 6.
2. Elements of the improved first order method

2.1. Admissible measures for first order local regularization

For \( \alpha \in \Lambda \), the definitions in Section 1 involve a signed measure \( \eta_\alpha \) given by

\[
d\eta_\alpha(\rho) := \frac{\lambda(\rho) \, dw_\alpha(\rho)}{\int_0^\alpha \lambda(s) \, dw_\alpha(s)} ,
\]

where \( \lambda \) is defined in (12) and \( \{w_\alpha\}_{\alpha \in \Lambda} \) is a family of admissible measures which we now define. We avoid the use of the positive measures \( w_\alpha \) employed in [9] and instead combine ideas therein with those from [1, 7] to establish the following definition. It is shown in (24) below that the denominator in (17) is nonzero for all \( \alpha \in \Lambda \).

**Definition 2.1.** A family \( \{w_\alpha\}_{\alpha \in \Lambda} \) of measures is said to be admissible for the first order local regularization of equation (1) if for every \( \alpha \in \Lambda \), \( w_\alpha \) is a finite signed measure defined on the Borel subsets of \([0, \alpha]\) such that the following properties are satisfied.

(i) There exists \( \sigma \in \mathbb{R} \) such that for each \( j = \nu + 1, \nu + 2, \ldots, 2\nu + 2 \),

\[
\int_0^\alpha \rho^j \, dw_\alpha(\rho) = \alpha^{j+\sigma} c_j (1 + O(\alpha)), \quad \text{as } \alpha \to 0,
\]

where the constants \( c_{\nu+1}, \ldots, c_{2\nu+2} \in \mathbb{R} \), \( c_{2\nu+2} \neq 0 \), are such that the roots of the polynomial \( p_\nu \) defined by

\[
p_\nu(t) := \sum_{\ell=0}^{\nu+1} \frac{c_{\nu+\ell+1}}{\ell!} \, t^\ell
\]

have negative real part. Without loss of generality, we let \( c_{2\nu+2} > 0 \).

(ii) There exists \( \tilde{C} > 0 \) such that for each \( \alpha \in \Lambda \),

\[
|w_\alpha|(0, \alpha] = \int_0^\alpha |w_\alpha|(\rho) \leq \tilde{C} \alpha^{\sigma},
\]

where \( |w_\alpha| \) denotes the total variation measure.

**Proposition 2.1.** Let \( c_{\nu+1}, c_{\nu+2}, \ldots, c_{2\nu+2} \in \mathbb{R} \), \( c_{2\nu+2} \neq 0 \), be such that the roots \( r_1, r_2, \ldots, r_{\nu+1} \in \mathbb{C} \) of the polynomial \( p_\nu \) defined in (18) satisfy \( \Re(r_j) < 0 \) for all \( j = 1, \ldots, \nu + 1 \). Then the following hold.

1. The coefficients \( c_{2\nu+1}, c_{\nu+1}, c_{2\nu+2} \) all have the same sign, and thus are positive under the standing assumption that \( c_{2\nu+2} > 0 \).
2. If all of the roots of \( p_\nu \) are real, then \( c_{2\nu} c_{2\nu+2} - c_{2\nu+1}^2 < 0 \).
Proof. Given \( p_\nu(t) = \sum_{j=0}^{\nu+1} \frac{c_{\nu+j+1}}{j!} t^j = \frac{c_{2\nu+2}}{(\nu+1)!} \prod_{j=1}^{\nu+1} (t - r_j) \), one has the relationship (see e.g. [12])
\[
    s_i = \frac{(-1)^i c_{2\nu+2-i} / (\nu + 1 - i)!}{c_{2\nu+2} / (\nu + 1)!}, \quad i = 1, \ldots, \nu + 1,
\]
that links the coefficients of \( p_\nu \) to the elementary symmetric polynomials, \( s_i = \sum_{1 \leq j_1, j_2, \ldots, j_i \leq n} r_{j_1} \cdots r_{j_i} \), \( i = 1, \ldots, \nu + 1 \), of its roots. In particular,
\[
    c_{\nu+1} = \frac{(-1)^{\nu+1} c_{2\nu+2} \prod_{j=1}^{\nu+1} r_j}{(\nu + 1)!}, \quad c_{2\nu} = \frac{c_{2\nu+2}}{\nu(\nu + 1)} \sum_{j=1}^{\nu} \sum_{k=j+1}^{\nu+1} r_j r_k,
\]
\[
    c_{2\nu+1} = -\frac{c_{2\nu+2}}{\nu + 1} \sum_{j=1}^{\nu+1} \Re(r_j).
\]
Since \( p_\nu \) has real coefficients and all roots have negative real part, it follows that \((-1)^{\nu+1} \prod_{j=1}^{\nu+1} r_j > 0\) and \(-\sum_{j=1}^{\nu+1} \Re(r_j) > 0\). Therefore both \( c_{\nu+1} \) and \( c_{2\nu+1} \) have the same sign as \( c_{2\nu+2} \).

A straightforward calculation yields \( s_1^2 = 2s_2 + \sum_{j=1}^{\nu+1} r_j^2 \), so that
\[
    c_{2\nu} c_{2\nu+2} - c_{2\nu+1}^2 = \left( \frac{c_{2\nu+2}^2}{\nu(\nu + 1)} \right) \left[ s_2 - \frac{\nu}{\nu + 1} s_1^2 \right] = \left( \frac{c_{2\nu+2}^2}{2\nu(\nu + 1)} \right) \left[ \nu - 1 \sum_{j=1}^{\nu+1} \Re(r_j)^2 + \sum_{j=1}^{\nu+1} (\Re(r_j)^2 - \Im(r_j)^2) \right].
\]

If all of the roots of \( p_\nu \) are real, then \( c_{2\nu} c_{2\nu+2} - c_{2\nu+1}^2 < 0 \), as the expression in brackets in (19) is strictly positive.

Unlike the situation for the positive measures, one can always find a family of measures satisfying Definition 2.1 for every \( \nu \in \mathbb{N} \). A method for the stabilized (when \( \beta > 0 \)) construction of an admissible family of measures is analogous to the one used in [1] for zeroth order local regularization of equation (1).

Lemma 2.1. For arbitrary negative real numbers \(-m_1, -m_2, \ldots, -m_{\nu+1}\), let \( \bar{c} = (\bar{c}_{\nu+1}, \bar{c}_{\nu+2}, \ldots, \bar{c}_{2\nu+2})^\top \in \mathbb{R}^{\nu+2} \) be defined by
\[
    p_\nu(t; \bar{c}) := \sum_{\ell=0}^{\nu+1} \frac{\bar{c}_{\nu+\ell+1}}{\ell!} t^\ell = \prod_{\ell=1}^{\nu+1} (t + m_\ell).
\]
For \( \beta \geq 0 \) and \( X \) the nonsingular \((\nu+2)\)-square submatrix of the Hilbert matrix that has entries \( X_{i,j} = 1/(\nu+i+j) \), let \( w(\beta) = (w_0(\beta), w_1(\beta), \ldots, w_{\nu+1}(\beta))^\top \in \mathbb{R}^{\nu+2} \) denote the unique solution of the problem
\[
    \min_{w \in \mathbb{R}^{\nu+2}} \left[ \| Xw - \bar{c} \|_2^2 + \beta \| w \|_2^2 \right],
\]
(20)
(a Tikhonov regularization problem in the Euclidean norm whenever $\beta > 0$). Then if $\beta \geq 0$ is sufficiently small, $w(\beta)$ defines a polynomial
\[\psi(\rho; \beta) := \sum_{\ell=0}^{\nu+1} w_\ell(\beta) \rho^\ell,\]
for which the $\beta$-dependent collection $\{w_\alpha\}_{\alpha \in \Lambda}$ of continuous measures defined via
\[dw_\alpha(\rho) := \psi\left(\frac{\rho}{\alpha}; \beta\right) d\rho, \quad \alpha \in \Lambda,\]
is admissible, with $c_{\nu+1}, \ldots, c_{2\nu+2}$ given by $c(\beta) = (c_{\nu+1}(\beta), \ldots, c_{2\nu+2}(\beta))^\top = Xw(\beta)$, $\sigma = 1$, and $\tilde{C} = \|\psi\|_{L^1([0,1])}$.

Proof. The result follows from the arguments in Lemma 3.1 of [1] with $\nu$ replaced by $\nu + 1$, the vector $(0!d_0, 1!d_1, \ldots, \nu!d_\nu)$ associated with $\bar{d} \in \mathbb{R}^{\nu+1}$ in [1] corresponding to $\bar{c} \in \mathbb{R}^{\nu+2}$ above, and changes in the entries in $X$ above due to the special form of (i) in Definition 2.1 in this section. Because the classical Hilbert matrix is totally positive, the square submatrix $X$ has a positive determinant.

\[\square\]

2.2. Parameter estimates

Let $\{w_\alpha\}_{\alpha \in \Lambda}$ be an admissible family of measures and let $\eta_\alpha$ be given by (17). We now provide estimates of the measure-dependent quantities defined in Section 1. Calculations are straightforward and follow immediately from Definition 2.1 and the assumed properties of $k$, namely
\[
k(t) = \frac{t^{\nu-1}}{(\nu-1)!} (1 + O(t)), \quad \text{as } t \to 0. \tag{21}\]

Starting with estimates directly pertinent to the measure $\eta_\alpha$, the parameter $\lambda$ defined in (12) satisfies
\[
\lambda(\rho) = \frac{\rho^{\nu+1}}{(\nu+1)!} (1 + O(\rho)), \quad \text{as } \rho \to 0, \tag{22}\]
which yields for $\gamma_\alpha$, the denominator in the definition (17) of $\eta_\alpha$, i.e.,
\[
\gamma_\alpha := \int_0^\infty \lambda(\rho) \, dw_\alpha(\rho), \tag{23}\]
the estimate
\[
\gamma_\alpha = \frac{c_{\nu+1}}{(\nu+1)!} \alpha^{\sigma+\nu+1} (1 + O(\alpha)), \quad \text{as } \alpha \to 0, \tag{24}\]
where \( c_{\nu+1} > 0 \) from Proposition 2.1. Thus, for all \( \alpha > 0 \) sufficiently small, the measure \( \eta_\alpha \) is well-defined with 
\[
\eta_\alpha([0, \alpha]) = 1 
\]
and
\[
|\eta_\alpha|([0, \alpha]) = \frac{1}{c_{\nu+1}} \int_0^\alpha \lambda(\rho) \, d\rho 
\]
with the upper bound in (25) satisfying 
\[ 2\tilde{C}/c_{\nu+1} \geq 1 \]
for all \( \alpha \) sufficiently small; indeed, for such \( \alpha \),
\[
\alpha^{j+\sigma} \frac{|c_j|}{2} \leq \left| \int_0^\alpha \rho^j \, dw_\alpha(\rho) \right| \leq \tilde{C} \alpha^{j+\sigma}, \quad j = \nu + 1, \ldots, 2\nu + 2,
\]
using Definition 2.1.

In addition, the parameters \( a_\alpha, b_\alpha \) defined by (11), (12), respectively, satisfy
\[
a_\alpha = \frac{c_{2\nu+1}}{c_{\nu+1} \nu!} \alpha^\nu (1 + \mathcal{O}(\alpha)), \quad (26)
\]
\[
b_\alpha = \frac{c_{2\nu+2}}{c_{\nu+1} (\nu + 1)!} \alpha^{\nu+1} (1 + \mathcal{O}(\alpha)), \quad (27)
\]
as \( \alpha \to 0 \), where \( c_{2\nu+1}, c_{2\nu+2} > 0 \) from Proposition 2.1 and Definition 2.1. We henceforth assume \( \tilde{\alpha} \) to be sufficiently small so that \( \eta_\alpha \) is well-defined and
\[ a_\alpha > 0, \quad b_\alpha > 0, \]
for all \( \alpha \in \Lambda \).

The \( 2 \times 2 \) system (13) that defines an approximate initial value \( u_0 \) relies on the parameters
\[
c_\alpha := \frac{1}{\gamma_\alpha} \int_0^\alpha \kappa^2(\rho) \, dw_\alpha(\rho), \quad \kappa(\rho) := \int_0^\rho k(s) \, ds, \quad \rho \in (0, \alpha], \quad (28)
\]
\[
\tilde{\delta}_\alpha := \frac{1}{\gamma_\alpha} \int_0^\alpha \tilde{\delta}(\rho) \kappa(\rho) \, dw_\alpha(\rho), \quad \alpha \in \Lambda. \quad (29)
\]
It is not difficult to show, for \( \rho \in (0, \alpha] \),
\[
\kappa(\rho) = \frac{\rho^\nu}{\nu!} (1 + \mathcal{O}(\alpha)) \quad \text{and} \quad c_\alpha = \frac{c_{2\nu}(\nu + 1)}{c_{\nu+1} \nu!} \alpha^{\nu-1} (1 + \mathcal{O}(\alpha)), \quad (30)
\]
as \( \alpha \to 0 \).

**Remark 2.1.** Many of quantities and estimates above correspond to those in [9], despite the change to a signed measure \( w_\alpha \) and the use of \( L^p(I) \) instead of \( C(I) \). We use notation that aligns with [1] as our development builds upon results obtained therein. The main notational changes from those in [9] to those appearing here are \( \tau \to \rho, \rho \to \alpha, \tilde{k}(t) \to \gamma_\alpha k_\alpha(t), \tilde{f}(t) \to \gamma_\alpha T_\alpha f(t), a \to \gamma_\alpha c_\alpha, b \to \gamma_\alpha a_\alpha, \) and \( c \to \gamma_\alpha b_\alpha \).

With these changes and a rescaling, our approximating equation (9) corresponds to equation (2.23b) in [9], while in (10) we allow for a more general approximate initial condition than the particular condition (2.23a) employed in that reference.
The next proposition characterizes properties of the operators $T_\alpha$, $D_\alpha$, and $A_\alpha$ defined in (3), (5), and (6), respectively.

**Proposition 2.2.** There exists $M > 0$ such that for all $\alpha \in \Lambda$, the operator $T_\alpha \in \mathcal{L}(L^p(I), L^p(I))$ and satisfies

$$\|T_\alpha g\|_{L^p(I)} \leq M \|g\|_{L^p(I)}, \quad g \in L^p(I).$$

Similarly, $D_\alpha \in \mathcal{L}(L^p(I), L^p(I_0))$ and $A_\alpha \in \mathcal{L}(L^p(I_0))$ with corresponding operator norms bounded uniformly in $\alpha \in \Lambda$.

**Proof.** Using arguments akin to those in [1, Proposition 3.1], the Minkowski inequality for integrals and (25) may be used to obtain, for each $\alpha \in \Lambda$ and $g \in L^p(I)$,

$$\|T_\alpha g\|_{L^p(I_0)} \leq \|g\|_{L^p(I)} |\eta_\alpha|(0, \alpha) \leq \frac{2\tilde{C}}{c_{\nu+1}} \|g\|_{L^p(I)}.$$

Boundedness of $D_\alpha$ is established in a similar way, after which the operator decomposition

$$D_\alpha + A_\alpha r = T_\alpha A,$$

(31)

derived from equation (4) with $f$ replaced by $Au$ can be used to conclude the boundedness of $A_\alpha$. \qed

### 2.3. Preliminary results

The main convergence results in Section 3 make use of the following two lemmas. To obtain a bound for later use, we include the value $p = 1$ in the first lemma (only).

**Lemma 2.2.** For every $\alpha \in \Lambda$, there exists a unique solution $\bar{z}_\alpha \in C^1(I_0)$ of

$$a_\alpha y(t) + b_\alpha y'(t) + \int_0^t k_\alpha(t-s)y(s) \, ds = 0, \quad t \in I_0,$$

(32)

$$y(0) = 1.$$  

(33)

Moreover, there exist constants $\hat{c} > 0$ (dependent on $\{w_\alpha\}_{\alpha \in \Lambda}$) and $C_z > 0$ (dependent on $k$ and $\{w_\alpha\}_{\alpha \in \Lambda}$) independent of $\alpha$ such that

(i) $|\bar{z}_\alpha(t)| \leq C_z$ for all $t \in I_0$, and

(ii) $\|\bar{z}_\alpha\|_{L^p(I_0)} \leq C_z \alpha^{1/p}$,

for all $p \in [1, \infty)$ whenever $\|k^{(\nu)}\|_{L^1(I_0)} < \hat{c}$.

**Proof.** Fix $\alpha \in \Lambda$. Existence of a unique $\bar{z}_\alpha \in C^1(I_0)$ solving (32)-(33) is a consequence of Proposition 3.1 in [9], where in our case we have a homogeneous equation and initial condition given explicitly.
As in [9], we make use of the transformation \( z_\alpha(t) := \bar{z}_\alpha(at), \ t \in [0, 1/\alpha], \) and note that \( z_\alpha \) satisfies equation (3.24) of that reference along with the initial condition \( z_\alpha(0) = 1. \) From Lemma 3.2 of [9]\(^1\) (cf. pp 2091, 2093), one has the bound \( |z_\alpha(t)| \leq Ke^{-ct} \) for some constants \( K, c > 0 \) independent of \( \alpha \) and \( p, \) and for all \( t \in [0, 1/\alpha], \) Thus \( |z_\alpha(t)| \leq C_z, \ t \in [0, 1/\alpha], \) and

\[
\|z_\alpha\|_{L^p([0,1/\alpha])} \leq \frac{K}{(cp)^{1/p}} \leq C_z
\]

for some \( C_z > 0, \) using the fact that \((cp)^{1/p} \geq \min\{1, c\}\) for \( p \in [1, \infty).\) Changing coordinates back to \( t \in [0, 1], \) the bounds in (i) and (ii) for \( \bar{z}_\alpha \) are obtained.

**Lemma 2.3.** Let \( h \in L^p(I_0). \) For each \( \alpha \in \Lambda, \)

\[
y_\alpha(t) := \frac{1}{b_\alpha} \int_0^t \bar{z}_\alpha(t-s)h(s)\, ds + y_0 \bar{z}_\alpha(t), \quad t \in I_0,
\]

is the unique solution in \( W^{1,p}(I_0) \) of the initial value problem

\[
a_\alpha y(t) + b_\alpha y'(t) + \int_0^t k_\alpha(t-s)y(s)\, ds = h(t), \quad \text{a.e. } t \in I_0,
\]

\[
y(0) = y_0.
\]

with

\[
\|y_\alpha\|_{L^p(I_0)} \leq \left( \frac{\alpha C_z}{b_\alpha} \right) \|h\|_{L^p(I_0)} + C_z \alpha^{1/p} |y_0|,
\]

whenever \( \|k^{(\nu)}\|_{L^1(I_0)} \leq \bar{c}, \) for \( \bar{z}_\alpha \) and constants \( \bar{c} > 0, \) \( C_z > 0 \) defined in Lemma 2.2. Further, the operator \( (a_\alpha L_\alpha + A_\alpha) : \text{dom}(L_\alpha) \subseteq L^p(I_0) \rightarrow L^p(I_0) \)

is injective with

\[
\left\| (a_\alpha L_\alpha + A_\alpha)^{-1} \right\| \leq \frac{\alpha C_z}{b_\alpha} = O \left( \frac{1}{\alpha^p} \right) \text{ as } \alpha \rightarrow 0,
\]

where \( \| \cdot \| \) denotes the \( \mathcal{L}(L^p(I_0)) \) operator norm.

**Proof.** For \( y_\alpha \) defined in (34), \( y_\alpha(0) = y_0. \) From the absolutely continuity of \( y_\alpha \) on \( I_0, \) it follows that

\[
b_\alpha y'_\alpha(t) = \bar{z}_\alpha(0)h(t) + \int_0^t \bar{z}_\alpha(t-s)h(s)\, ds + b_\alpha \bar{z}_\alpha(t)y_0
\]

\[
= h(t) + \int_0^t \left( \frac{-a_\alpha}{b_\alpha} \bar{z}_\alpha(t-s) - \frac{1}{b_\alpha} \int_0^{t-s} k_\alpha(t-s-\tau) \bar{z}_\alpha(\tau) \, d\tau \right) h(s) \, ds
\]

\[
- y_0 \left( a_\alpha \bar{z}_\alpha(t) + \int_0^t k_\alpha(t-\tau) \bar{z}_\alpha(\tau) \, d\tau \right)
\]

\[
= h(t) - a_\alpha y_\alpha(t) - \int_0^t k_\alpha(t-s)y_\alpha(s) \, ds,
\]

\(^1\)Only assumptions A1 and A3 from that reference are needed here.
a.e. $t \in I_0$. Hence $y_0 \in W^{1,p}(I_0)$ satisfies the initial value problem (35)-(36), and uniqueness of $y_α$ follows from the uniqueness of $\bar{z}_α$.

The bound on $\bar{z}_α$ in Lemma 2.2 along with the representation (34) for $y_α$ and Young’s inequality permit

$$\|y_α\|_{L^p(I_0)} \leq \frac{1}{b_α} \|\bar{z}_α\|_{L^1(I_0)}\|h\|_{L^p(I_0)} + \|\bar{z}_α y_0\|_{L^p(I_0)},$$

from which the bound in (37) obtains.

The remainder of the lemma follows from the definitions (6), (11), (16), of $A_α$, $a_α$, $L_α$, respectively, and the use of $y_0 = 0$ in equations (35)–(36). □

3. Main Convergence Results

Let $\bar{u} := A^{-1}f$ denote the solution of equation (1) and henceforth assume that $\bar{u} \in W^{1,p}(I)$. In order to further characterize the regularity of $\bar{u}$, we make the following definition

$$\mu(\bar{u}, α) := \frac{1}{α} \int_0^α |\bar{u}'(\cdot + ξ) - \bar{u}'(\cdot)| dξ \|_{L^p(I_0)}.$$  \hspace{1cm} (39)

We proceed to show that the initial value problem (9)–(10) is well-posed with regularized solution $u_δ^α \in W^{1,p}(I_0)$, and that $u_δ^α$ is convergent to $r\bar{u}$ in $L^p(I_0)$ under suitable conditions on $α, δ \to 0$, and under the convergence of the approximate initial value $u_0 = u_0(α, f^δ)$ to $\bar{u}(0)$.

Our main theorem is stated as follows. In Section 3.1 below, we establish that either of the approximate initial values $u_0(α, f^δ)$ defined in (13) or (14) satisfy the conditions of this theorem.

**Theorem 3.1.** Let $\{w_α\}_{α \in Λ}$ be an admissible family of measures. For each $α \in Λ$, let $η_α$ be defined by (17) and suppose that the mapping $f^δ \to u_0 = u_0(α, f^δ)$ is continuous from $L^p(I)$ to $\mathbb{R}$. Then for every $α \in Λ$, there is a unique solution $u_δ^α \in W^{1,p}(I_0)$ of the first order local regularization equations (9)–(10) that depends continuously on $f^δ \in L^p(I)$ whenever $\|k^{(ν)}\|_{L^1(I_0)} < \hat{c}$, where $\hat{c} > 0$ denotes the constant defined in Lemma 2.2.

Furthermore, if $α = α(δ) \to 0$ as $δ \to 0$ and there is some function $ω_0 : Λ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that, for such $α(δ)$,

$$|u_0(α, f^δ) - \bar{u}(0)| = O(ω_0(α, δ)) \text{ as } δ \to 0,$$ \hspace{1cm} (40)

then

$$\|u_δ^α - r\bar{u}\|_{L^p(I_0)} = O\left(\frac{δ}{α^ν} + ω_0 u(α, α) + α^{1/p}ω_0(α, δ)\right) \text{ as } δ \to 0,$$ \hspace{1cm} (41)

where $μ$ in (39) satisfies $μ(\bar{u}, α) \to 0$ as $δ \to 0$. 

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Proof. Suppose the hypotheses of the theorem and fix \( \alpha \in \Lambda \). From the assumed continuity of the mapping \( \delta \mapsto u_0(\alpha, \delta) \in \mathbb{R} \), it follows from Lemma 2.3 that the unique solution \( u_\alpha^\delta = u_\alpha^\delta(\cdot; u_0) \in L^p(I_0) \) of (9)–(10) depends continuously on data \( \delta \in L^p(I) \).

Further, if \( u_\alpha^\delta(\cdot; \bar{u}(0)) \in W^{1,p}(I_0) \) denotes the solution of (9)–(10) with initial value \( \bar{u}(0) \), then

\[
\|u_\alpha^\delta(\cdot; u_0) - r\bar{u}\|_{L^p(I_0)} 
\leq \|u_\alpha^\delta(\cdot; u_0) - u_\alpha^\delta(\cdot; \bar{u}(0))\|_{L^p(I_0)} + \|u_\alpha^\delta(\cdot; \bar{u}(0)) - r\bar{u}\|_{L^p(I_0)} 
\leq C_\alpha \alpha^{1/p}|u_0(\alpha, f^\delta) - \bar{u}(0)| + \|v_\alpha^t - r\bar{v}\|_{L^p(I_0)} \tag{42}
\]

from Lemma 2.3, where we make use of two translations

\[
v_\alpha^\delta := u_\alpha^\delta(\cdot; \bar{u}(0)) - \bar{u}(0)r1, \\
\bar{v} := \bar{u} - \bar{u}(0)1,
\]

defined to ensure \( v_\alpha^\delta(0) = \bar{v}(0) = 0 \). Here, 1 denotes the function whose value is identically one on \( I \). Note that \( v_\alpha^\delta \in \text{dom}(L_\alpha) \subset L^p(I_0) \) is the unique solution of

\[
(a_\alpha L_\alpha + A_\alpha)v = g_\alpha^\delta, \\
g_\alpha^\delta := T_\alpha f^\delta - \bar{u}(0)(a_\alpha r + A_\alpha r)1, \tag{43}
\]

while \( \bar{v} \in W^{1,p}(I) \) solves the equation \( A\bar{v} = f - \bar{u}(0)A1 \), or alternatively,

\[
T_\alpha A\bar{v} = g_\alpha, \\
g_\alpha := T_\alpha f - \bar{u}(0)T_\alpha A1. \tag{44}
\]

Employment of the decomposition of \( T_\alpha A \) in (31) leads to the observation that \( \bar{v} \) also satisfies

\[
(a_\alpha L_\alpha + A_\alpha)\bar{v} = g_\alpha - (D_\alpha - a_\alpha L_\alpha r)\bar{v}. \tag{45}
\]

It follows then from (43) and (45) that

\[
v_\alpha^\delta - r\bar{v} = (a_\alpha L_\alpha + A_\alpha)^{-1} \left[(g_\alpha^\delta - g_\alpha) + (D_\alpha - a_\alpha L_\alpha r)v\right] \tag{46}
\]

where

\[
g_\alpha^\delta - g_\alpha = T_\alpha (f^\delta - f) + \bar{u}(0)[T_\alpha A - a_\alpha r - A_\alpha r]1 = T_\alpha (f^\delta - f) + \bar{u}(0)[D_\alpha - a_\alpha r]1 = T_\alpha (f^\delta - f),
\]

giving

\[
\|g_\alpha^\delta - g_\alpha\|_{L^p(I)} \leq M\delta. \tag{47}
\]

Regarding the final term in (46), we proceed to show that

\[
\|(D_\alpha - a_\alpha L_\alpha r)v\|_{L^p(I_0)} \leq C\alpha^{p+1}\mu(\bar{v}, \alpha) \tag{48}
\]

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for some $C > 0$. For $\alpha$ sufficiently small and a.e. $t \in I_0$, and using the estimates (21), (25) on $k$, $|\eta_\alpha|$, respectively,

$$|(D_\alpha - a_\alpha L_\alpha r)\bar{v}(t)|$$

$$\leq \int_0^\alpha \left| \int_0^\rho k(\rho - s) (\bar{v}(t + s) - \bar{v}(t) - s\bar{v}'(t)) \, ds \right| \, d|\eta_\alpha|(\rho)$$

$$\leq \frac{2\tilde{C}}{c_{\nu + 1}} \|k\|_{C[0,\alpha]} \int_0^\alpha |\bar{v}(t + s) - \bar{v}(t) - s\bar{v}'(t)| \, ds$$

$$= \frac{2\tilde{C}}{c_{\nu + 1}} \|k\|_{C[0,\alpha]} \int_0^\alpha \left| \int_t^{t+s} (\bar{v}'(\xi) - \bar{v}'(t)) \, d\xi \right| \, ds$$

$$\leq C\alpha^\nu \int_0^\alpha |\bar{v}'(t + s) - \bar{v}'(t)| \, ds. \quad (49)$$

Therefore the bound in (48) obtains. As $\bar{v}' \in L^p(I)$, we follow the arguments in Lemma 3.4 of [1] that invoke the Lebesgue differentiation theorem and the Hardy-Littlewood maximal theorem for $p \in (1, \infty)$, to conclude

$$\lim_{\alpha \to 0} \mu(\bar{v}, \alpha) = \lim_{\alpha \to 0} \left( \int_0^1 \left( \frac{1}{\alpha} \int_0^\alpha |\bar{v}'(t + s) - \bar{v}'(t)| \, ds \right)^p \, dt \right)^{1/p} = 0.$$  

Finally, we return to (46) and apply (47), (48), and the bound (38) on the operator norm of $(a_\alpha L_\alpha + A_\alpha)^{-1}$ to obtain

$$\|v_\alpha - r\bar{v}\|_{L^p(I_0)} = O \left( \frac{\delta}{\alpha^\nu} + \alpha \mu(\bar{v}, \alpha) \right),$$

so that, with (42) and the fact that $\mu(\bar{u}, \alpha) = \mu(\bar{v}, \alpha)$, the statement of convergence in (41) follows. 

3.1. Convergence using approximate initial values

We now consider the issue of convergence when using one of the two approximate initial values $u_0 = u_0(\alpha, f^\delta)$ given by (13) (as defined in [9]) or (14).

We refer to the first component of the solution of the $2 \times 2$ system (13) as a first-order approximate initial value. Note that the second component in vector equation (13) is the same as equation (9) evaluated at $t = 0$ and therefore not needed; nevertheless the pairing in (13) allows one to write an explicit formula for $u_0$ (cf. (51) below) provided the family $\{w_\alpha\}_{\alpha \in \Lambda}$ satisfies the additional condition (cf. (A2) in [9])

$$c_2 \nu c_{2\nu + 2} - c_{2\nu + 1}^2 \neq 0. \quad (50)$$

When using this first-order approximate initial value, we implicitly assume that (50) holds. Fortunately, in light of Proposition 2.1 and Lemma 2.1, we may always construct an admissible family $\{w_\alpha\}_{\alpha \in \Lambda}$ in a stable manner for which (50) holds by simply taking the polynomial $p_\nu$ defined in (18) to have negative real roots.
Lemma 3.1. [9] Let \( \{ w_\alpha \}_{\alpha \in \Lambda} \) be an admissible family of measures which additionally satisfies the condition (50). Then \( c_\alpha b_\alpha - a_\alpha^2 \neq 0 \) and there is a unique \( u_0(\alpha, f^\delta) = u_\alpha(0) \) satisfying (13) given by
\[
u \left( T^{\nu+1/2} + (2c_\alpha^2 c_{2\nu+2} - c_{2\nu+1}^2) (1 + \mathcal{O}(\alpha)) \right) \neq 0
\]
(52)
as \( \alpha \to 0 \). The remainder of the lemma follows from an application of Cramer's rule.

Proof. Using (26), (27), and (30), it follows that
\[

(51)
\]
as \( \alpha \to 0 \). The remainder of the lemma follows from an application of Cramer's rule.

For the zeroth-order approximate initial value \( u_0 \) in (14), condition (50) is not needed. In the following theorem, we obtain a convergence rate in \( L^p(I_0) \) for either value of \( u_0 \) by choosing the measure to have a special form, and arrive at the same rate of convergence in each case. Define
\[
\varphi(\bar{u}, \alpha) := \max_{\tau \in (0, \alpha]} |\bar{u}(\tau) - \bar{u}(0)|.
\]
(53)

Theorem 3.2. Let \( \psi \) be a polynomial such that the family of measures \( \{ w_\alpha \}_{\alpha \in \Lambda} \) defined by
\[
dw_\alpha(\rho) := \psi \left( \frac{\rho}{\alpha} \right) \, d\rho, \quad \alpha \in \Lambda
\]
is admissible with \( \sigma = 1 \). Let \( u_\alpha^\delta \) be the solution of (9)-(10) where \( u_0 = u_0(\alpha, f^\delta) \) is a first-order or zeroth-order approximate initial value given by (51) or (14), respectively. Then the mapping \( f^\delta \mapsto u_0(\alpha, f^\delta) \) is continuous from \( L^p(I_0) \) to \( \mathbb{R} \), and if \( \alpha = \alpha(\delta) \to 0 \) as \( \delta \to 0 \), then
\[
|u_0 - \bar{u}(0)| = \mathcal{O} \left( \frac{\delta}{\alpha^{\nu+1/p}} + \varphi(\bar{u}, \alpha) \right) \quad \text{as} \ \delta \to 0,
\]
(54)
whenever \( \|k^{(\nu)}\|_{L^1(I_0)} < \hat{c} \), where \( \hat{c} > 0 \) denotes the constant defined in Lemma 2.2. Therefore, for such \( \alpha(\delta) \),
\[
\|u_\alpha^\delta - \tau \bar{u}\|_{L^p(I_0)} = \mathcal{O} \left( \frac{\delta}{\alpha^{\nu}} + \alpha \mu(\bar{u}, \alpha) + \alpha^{1/p} \varphi(\bar{u}, \alpha) \right) \quad \text{as} \ \delta \to 0.
\]
(55)
Proof. Fix \( \alpha \in \Lambda \). Note first that for any \( h \in L^p(I_0) \) and for \( q \) given by \( \frac{1}{p} + \frac{1}{q} = 1 \), we have the preliminary estimate
\[
\left| \int_0^\alpha h(\rho) \, dw_\alpha(\rho) \right| = \alpha \left| \int_0^1 h(\alpha \rho) \psi(\rho) d\rho \right| \leq \alpha^{1-1/p} \|h\|_{L^p(I_0)} \|\psi\|_{L^q(I_0)}.
\]
(56)
by a change of variables and Hölder’s inequality. Similarly,
\[
\left| \int_0^\alpha \int_0^\rho k(s) [\bar{u}(\rho - s) - \bar{u}(0)]ds \, dw_\alpha(\rho) \right| \leq \alpha^2 \|k\|_{C([0,\alpha])} \varphi(\bar{u}, \alpha) \|\psi\|_{L^1(I_\alpha)}. \tag{57}
\]

Suppose now that \( u_0 \) is a first-order approximate initial value. We first establish continuity of the mapping \( f^\delta \to u_0 \). Note that
\[
|u_0| = \frac{\gamma_\alpha}{\gamma_\alpha |c_\alpha b_\alpha - a_\alpha^2|} |b_\alpha g_\alpha^\delta - a_\alpha T_\alpha f^\delta(0)|,
\]
and
\[
\gamma_\alpha |b_\alpha g_\alpha^\delta - a_\alpha T_\alpha f^\delta(0)| = \alpha \left| \int_0^1 f^\delta(\alpha \rho) (b_\alpha \kappa(\alpha \rho) - a_\alpha \lambda(\alpha \rho)) \psi(\rho) d\rho \right|,
\]
then by (56),
\[
|u_0| \leq (d_1(\alpha) + d_2(\alpha)) \alpha^{1-1/p} \|\psi\|_{L^p(I_\alpha)} \|f^\delta\|_{L^p(I)},
\]
where
\[
d_1(\alpha) := \frac{\kappa(\alpha)}{\gamma_\alpha} \frac{b_\alpha}{|c_\alpha b_\alpha - a_\alpha^2|} \leq \frac{3c_{2k+2}}{c_{2k}c_{2k+2} - c_{2k+1}^2} \frac{\nu!}{\alpha^{\sigma+\nu}}, \tag{58}
\]
\[
d_2(\alpha) := \frac{\lambda(\alpha)}{\gamma_\alpha} \frac{a_\alpha}{|c_\alpha b_\alpha - a_\alpha^2|} \leq \frac{3c_{2k+1}}{c_{2k}c_{2k+2} - c_{2k+1}^2} \frac{\nu!}{\alpha^{\sigma+\nu}}, \tag{59}
\]
from (22), (26), (27), (30) and (52) for \( \alpha \) sufficiently small. Therefore, with \( \sigma = 1 \),
\[
|u_0| \leq \frac{3(c_{2k+2} + c_{2k+1})}{c_{2k}c_{2k+2} - c_{2k+1}^2} \frac{\nu!}{\alpha^{\nu+1/p}} \|\psi\|_{L^p(I_\alpha)} \|f^\delta\|_{L^p(I)}.
\]

Further,
\[
|u_0 - \bar{u}(0)| = \frac{\gamma_\alpha}{\gamma_\alpha |c_\alpha b_\alpha - a_\alpha^2|} |b_\alpha g_\alpha^\delta - a_\alpha T_\alpha f^\delta(0) - (c_\alpha b_\alpha - a_\alpha^2) \bar{u}(0)|, \tag{60}
\]
where
\[
\gamma_\alpha |g_\alpha^\delta - c_\alpha \bar{u}(0)| = \left| \int_0^\alpha (f^\delta(\rho) - f(\rho)) \kappa(\rho) \, dw_\alpha(\rho) \right. \\
+ \left. \int_0^\alpha \int_0^\rho k(s) (\bar{u}(\rho - s) - \bar{u}(0))ds \, \kappa(\rho) \, dw_\alpha(\rho) \right|,
\]
and
\[
\gamma_\alpha a_\alpha \left| \frac{T_\alpha f^\delta(0)}{a_\alpha} - \bar{u}(0) \right| = \left| \int_0^\alpha (f^\delta(\rho) - f(\rho)) \lambda(\rho) \, dw_\alpha(\rho) \right. \\
+ \left. \int_0^\alpha \int_0^\rho k(s) (\bar{u}(\rho - s) - \bar{u}(0))ds \, \lambda(\rho) \, dw_\alpha(\rho) \right|, \tag{61}
\]

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Use of the bounds in (56) and (57) gives
\[
|u_0 - \bar{u}(0)| \leq (d_1(\alpha) + d_2(\alpha)) \|\psi\|_{L^q(I_0)} \left( \alpha^{1-1/p} \delta + \alpha^2 \|k\|_{C([0,\alpha])} \varphi(\bar{u}, \alpha) \right)
\]
\[
\leq \frac{3(c_{2\nu+2} + c_{2\nu+1})!}{c_{2\nu+2} - c_{2\nu+1}} \|\psi\|_{L^q(I_0)} \left( \frac{\delta}{\alpha^{1/p}} + \frac{2}{(\nu - 1)!} \varphi(\bar{u}, \alpha) \right),
\]
for \( \bar{\alpha} \) sufficiently small, and so if \( \alpha = \alpha(\delta) \to 0 \) as \( \delta \to 0 \), we arrive at (54).

In the case that \( u_0 \) is a zeroth-order approximate initial value, we may argue similarly starting from (61). By the estimate
\[
\lambda(\alpha) \leq \frac{3\nu!}{\gamma_\alpha a_\alpha}
\]
it follows that
\[
|u_0 - \bar{u}(0)| \leq \frac{3\nu!}{c_{2\nu+1} \alpha^{\nu+\sigma}} \|\psi\|_{L^q(I_0)} \left[ \frac{\delta}{\alpha^{1/p}} + \frac{2}{(\nu - 1)!} \varphi(\bar{u}, \alpha) \right],
\]
for \( \bar{\alpha} \) sufficiently small which also leads to the rate in (54).

3.2. Rate of convergence

With the bound on \( \|u^{\delta}_{\alpha(\delta)} - r\bar{u}\|_{L^p(I_0)} \) established in (55) as a starting point, we determine below a rate for the convergence of \( u^{\delta}_{\alpha} \) to \( r\bar{u} \) under suitable source conditions on \( \bar{u} \).

Regarding the quantities \( \mu(\bar{u}, \alpha) \), \( \varphi(\bar{u}, \alpha) \), defined in (39) and (53), respectively, the standing assumption that \( \bar{u} \in W^{1,p}(I) \) gives
\[
\mu(\bar{u}, \alpha) \to 0, \quad \varphi(\bar{u}, \alpha) = O(\alpha), \quad \text{as } \alpha \to 0,
\]
which is insufficient to give a rate for the combined last two terms that appear in (55). The additional condition that \( \bar{u}' \) be Hölder continuous with exponent \( \zeta \in (0, 1) \) yields \( \mu(\bar{u}, \alpha) = O(\alpha^{\zeta}) \) as \( \alpha \to 0 \) with no improvement in the rate for \( \varphi(\bar{u}, \alpha) \); in this case, the last two terms in (55) are of order
\[
\alpha \mu(\bar{u}, \alpha) + \alpha^{1/p} \varphi(\bar{u}, \alpha) = O(\alpha^{\zeta+1}) + O \left( \alpha^{1/p+1} \right), \quad \text{as } \alpha \to 0.
\]
Thus the relative size of \( \zeta \) compared to \( 1/p \) determines the dominating term in (62).

However, if we add the condition that \( \bar{u}'(0) = 0 \), the assumption of Hölder continuity of \( \bar{u}' \) becomes useful in establishing an improved rate on \( \varphi \), namely,
\[
\varphi(\bar{u}, \alpha) \leq \max_{\tau \in (0,\alpha]} \int_0^\tau |\bar{u}'(\xi) - \bar{u}'(0)| \, d\xi = O(\alpha^{\zeta+1}), \quad \text{as } \alpha \to 0.
\]
The combined rate of the last two terms in (55) then leads to
\[
\alpha \mu(\bar{u}, \alpha) + \alpha^{1/p} \varphi(\bar{u}, \alpha) = O(\alpha^{\zeta+1}), \quad \text{as } \alpha \to 0.
\]
We may now balance all three terms in (55) with a particular choice of \( \alpha = \alpha(\delta) \) to obtain an overall rate of convergence of \( u^{\delta}_{\alpha(\delta)} \) to \( r\bar{u} \).
Corollary 3.1. Assume that the hypotheses of Theorem 3.2 hold with \( \tilde{u} \in \{ u \in W^{1,p}(I) \mid u'(0) = 0, u' \text{ is H"older continuous on } I \text{ with exponent } \zeta \in (0,1) \} \). Then for the choice \( \alpha(\delta) = \tilde{C}\delta^{-\frac{\zeta}{\zeta+1}} \) for \( \tilde{C} > 0 \), it holds that
\[
\|w^\delta_{\alpha(\delta)} - r\tilde{u}\|_{L^p(I_0)} = O\left(\delta^{-\frac{\zeta}{\zeta+1}}\right) \text{ as } \delta \to 0.
\]

4. First order regularization in the space of continuous functions

The context in [9] for the first order regularization of equation (1) is the space of continuous functions for both solution and data. In this section, we briefly return to this setting to compare our findings with those of [9]. As demonstrated below, all of the results for \( L^p \) spaces established in the previous sections carry over to the space of continuous functions with only minor changes.

To this end, let \( f \in C(I) \) and make the standing assumptions that \( \tilde{u} \in C^1(I) \) and that measured data \( f^\delta \in C(I) \) satisfies
\[
\|f - f^\delta\|_{C(I)} \leq \delta.
\]

Note that the operator \( A \) defined in (2) belongs to \( L(C(I)) \), and the governing equation (1) remains ill-posed in this setting (see, e.g., [6]).

With regard to the development of the first order local regularization equations in \( C(J_0) \), the definition of admissible measures in Section 2.1 still applies. However, in addition to an easily constructed continuous measure \( w_\alpha \) (Lemma 2.1), one also has the possibility of a discrete measure of the form, for \( \alpha \in \Lambda \),
\[
\int_0^\alpha g(\rho) \, dw_\alpha(\rho) := \sum_{\ell=0}^L w_\ell g(\tau_\ell\alpha), \; g \in C[0,\alpha],
\]
provided that the weights \( w_\ell \) and nodes \( \tau_\ell \in [0,1], \; \ell = 0, \ldots, L, \) are selected to satisfy the conditions in Definition 2.1. The derivation of one such measure follows.

Lemma 4.1. Fix arbitrary negative real numbers \( -m_1, -m_2, \ldots, -m_{\nu+1} \) and let \( \tau_\ell, \; \ell = 0, \ldots, \nu + 1, \) satisfy \( 0 < \tau_0 < \tau_1 < \cdots < \tau_{\nu+1} \leq 1 \). Let \( \bar{c} = (\bar{c}_{\nu+1}, \bar{c}_{\nu+2}, \ldots, \bar{c}_{2\nu+2})^T \in \mathbb{R}^{\nu+2} \) and \( w(\beta) = (w_0(\beta), w_1(\beta), \ldots, w_{\nu+1}(\beta))^T \in \mathbb{R}^{\nu+2} \) be as defined in Lemma 2.1 for \( \beta \geq 0 \), where \( X \) in (20) is the \((\nu+2)\)-square matrix that has entries \( X_{i,j} = \tau_j^{-\frac{3}{2}} \).

Then if \( \beta \geq 0 \) is sufficiently small, the \( \beta \)-dependent collection \( \{ w_\alpha \}_{\alpha \in \Lambda} \) of discrete measures defined via
\[
\int_0^\alpha g(\rho) \, dw_\alpha(\rho) := \sum_{\ell=0}^{\nu+1} w_\ell(\beta) g(\tau_\ell\alpha),
\]
for \( \alpha \in \Lambda, \; g \in C[0,\alpha], \) is admissible, with \( c_{\nu+1}, \ldots, c_{2\nu+2} \) given by \( c(\beta) = (c_{\nu+1}(\beta), \ldots, c_{2\nu+2}(\beta))^T = Xw(\beta), \) \( \sigma = 0, \) and \( \bar{C} = \|w(\beta)\|_1.\)
Theorem 4.1. Let \( T_\alpha, D_\alpha, r \in \mathcal{L}(C(I), C(I_0)) \), \( A_\alpha \in \mathcal{L}(C(I_0)) \), \( \text{dom}(L_\alpha) = \{ u \in C^1(I_0), u(0) = 0 \} \), and with the same bound (38) on the \( \mathcal{L}(C(I_0)) \) operator norm \( \| \cdot \| \) applied to \( (a_\alpha L_\alpha + A_\alpha)^{-1} \). In addition, using Lemma 2.2 and arguments like those from the proof of Lemma 2.3, the unique solution of (35)–(36) given by \( y_\alpha \) in (34) satisfies

\[
\| y_\alpha \|_{C(I_0)} \leq \frac{1}{b_\alpha} \| \tilde{z}_\alpha \|_{L^1(I_0)} \| h \|_{C(I)} + \| \tilde{z}_\alpha \|_{C(I_0)} |y_0| \\
\leq \frac{\alpha C_\varepsilon}{b_\alpha} \| h \|_{C(I)} + C_\varepsilon |y_0|.
\]  

(64)

Convergence of \( u^\delta_\alpha \) to \( r\bar{u} \) holds with slight changes from the \( L^p \) case (Section 3), now making use of an alternate definition of \( \mu \), namely,

\[
\mu(\bar{u}, \alpha) := \sup_{t \in I_0} \{ |\bar{u}'(t + \xi) - \bar{u}'(t)|, \xi \in [0, \alpha] \}.
\]

(65)

Theorem 4.1. Let \( \{ w_\alpha \}_{\alpha \in \Lambda} \) denote an admissible family of measures. For each \( \alpha \in \Lambda \), let \( \eta_\alpha \) be defined by (17) and suppose that the mapping \( f^\delta \mapsto u_\alpha(\cdot, f^\delta) \) is continuous from \( C(I) \) to \( \mathbb{R} \). Then for every \( \alpha \in \Lambda \), there is a unique solution \( u^\delta_\alpha \in C^1(I_0) \) of the first order local regularization equations (9)–(10) that depends continuously on \( f^\delta \in C(I) \) whenever \( \| k^{(\alpha)} \|_{L^1(I_0)} < \hat{c} \), where \( \hat{c} > 0 \) denotes the constant defined in Lemma 2.2.

Furthermore, if \( \alpha = \alpha(\delta) \to 0 \) as \( \delta \to 0 \) and there is some function \( \omega_0 : \Lambda \times \mathbb{R}^+ \to \mathbb{R}^+ \) such that (40) holds, then

\[
\| u^\delta_\alpha - r\bar{u} \|_{C(I_0)} = O \left( \frac{\delta}{\alpha^{\hat{c}}} + \alpha \mu(\bar{u}, \alpha) + \omega_0(\alpha, \delta) \right) \quad \text{as} \ \delta \to 0,
\]

(66)

where \( \mu \) in (65) satisfies \( \mu(\bar{u}, \alpha) \to 0 \) as \( \delta \to 0 \).

Proof. Well-posedness of equations (9)–(10) follows from Lemma 2.3 and (64). We may follow the arguments in the proof of Theorem 3.1 and employ the same definitions for \( u^\delta_\alpha(\cdot; \bar{u}(0)), u^\delta_\alpha(\cdot; u_0), v^\delta_\alpha \), and \( \bar{v} \), to find

\[
\| u^\delta_\alpha(\cdot; u_0) - r\bar{u} \|_{C(I_0)} \leq \| u^\delta_\alpha(\cdot; u_0) - u^\delta_\alpha(\cdot; \bar{u}(0)) \|_{C(I_0)} + \| u^\delta_\alpha(\cdot; \bar{u}(0) - r\bar{v} \|_{C(I_0)} \\
\leq C_\varepsilon \| u_0(\alpha, f^\delta) - \bar{u}(0) \| + \| v^\delta_\alpha - r\bar{v} \|_{C(I_0)} \\
\leq C_\varepsilon \omega_0(\alpha, \delta) + \| (a_\alpha L_\alpha + A_\alpha)^{-1} \left( M\delta + \| (D_\alpha - a_\alpha L_\alpha r) \bar{v} \|_{C(I_0)} \right),
\]

(65)
with the use of (46) and (47). Then \(\|(D_\alpha - a_\alpha L_\alpha r)\vec{v}\| \leq C \alpha^{\nu+1} \mu(\vec{u}, \alpha)\) follows from (49) and (65), and use of (38) results in the desired bound (66). 

The same constructions for first-order and zeroth-order approximate initial values are applicable in the case of continuous data \(f^\delta\).

**Theorem 4.2.** Let \(\{w_\alpha\}_{\alpha \in \Lambda}\) denote an admissible family of measures, and for each \(\alpha \in \Lambda\), let \(\eta_\alpha\) be defined by (17). Let \(u_0^\delta\) be the solution of (9)-(10) where \(u_0 = u_0(\alpha, f^\delta)\) is a first-order or zeroth-order approximate initial value given by (51) or (14), respectively. Then the mapping \(f^\delta \mapsto u_0(\alpha, f^\delta)\) is continuous from \(C(I)\) to \(\mathbb{R}\), and if \(\alpha = \alpha(\delta) \to 0\) as \(\delta \to 0\), then

\[
|u_0 - \bar{u}(0)| = \mathcal{O}\left(\frac{\delta}{\alpha^\nu} + \varphi(\bar{u}, \alpha)\right) \quad \text{as } \delta \to 0, \tag{67}
\]

whenever \(\|k^{(\nu)}\|_{L^1(I_0)} < \hat{c}\), where \(\hat{c} > 0\) denotes the constant defined in Lemma 2.2. Therefore, for such \(\alpha(\delta)\),

\[
\|u_0^\delta - r\bar{u}\|_{C(I_0)} = \mathcal{O}\left(\frac{\delta}{\alpha^\nu} + \alpha \mu(\bar{u}, \alpha) + \varphi(\bar{u}, \alpha)\right) \quad \text{as } \delta \to 0.
\]

**Proof.** For \(\alpha \in \Lambda\), we observe that the mapping \(f^\delta \mapsto u_0(\alpha, f^\delta)\) from \(C(I)\) to \(\mathbb{R}\) is continuous for both initial values. For the first-order approximate initial value, the estimates in (60)–(61) give that

\[
|u_0 - \bar{u}(0)| \leq \hat{C} \alpha^\sigma (d_1(\alpha) + d_2(\alpha)) \left(\delta + \alpha \|k\|_{C[0,\alpha]} \varphi(\bar{u}, \alpha)\right)
\]

while for the zeroth-order approximate, we use (61) to argue

\[
|u_0 - \bar{u}(0)| \leq \hat{C} \alpha^\sigma d_2(\alpha) \left(\delta + \alpha \|k\|_{C[0,\alpha]} \varphi(\bar{u}, \alpha)\right),
\]

so that, using (58)–(59), estimate (67) holds in both cases. The remainder of the proof follows from Theorem 4.1. 

In the space of continuous functions, we obtain the same overall convergence rate as in Corollary 3.1.

**Corollary 4.1.** Assume that the hypotheses of Theorem 4.2 hold with \(\bar{u} \in \{u \in C^1(I) \mid u'(0) = 0, u' \text{ is Hölder continuous on } I \text{ with exponent } \zeta \in (0,1]\}\). Then for the choice \(\alpha(\delta) = \bar{C} \delta^{-\frac{1}{\zeta+1}}\) for \(\bar{C} > 0\), it holds that

\[
\|u_0^\delta - r\bar{u}\|_{C(I_0)} = \mathcal{O}\left(\delta^{-\frac{1}{\zeta+1}}\right) \quad \text{as } \delta \to 0.
\]

**Remark 4.1.** While a comparison might lead one to conclude that we obtain an improved convergence rate over that in [9], in fact, there is a small error in [9] which, when corrected, yields the same overall result. Indeed, a factor of \(\rho\) is missing from the right-hand side of equation (4.12) of [9]; once that factor is reinstated, an additional argument at the end of the proof of Theorem 4.1 in [9] leads to the same result as in Theorem 4.2 above.
5. Numerical Examples

To compare with the numerical examples that appear in [9], we assume that the original problem (1) is set in the space of continuous functions on the interval $[0, T]$ for some $T > 0$ with $\nu$-smoothing kernel $k(t) = t^{\nu - 1}$ for $\nu \in \mathbb{N}$, and note that the theory in Sections 1–4 carries over with trivial changes. As in [9], we perform collocation by continuous piecewise linear functions to solve the first order local regularization equations (9)–(10) numerically. A discrete measure is still used, only now we permit signed weights to satisfy Definition 2.1.

Our examples display reconstructed solutions based on exact and noisy data for values of $\nu \geq 3$. They demonstrate the convergence and stability of our method as established in Theorem 4.1 and the rate of convergence suggested in Corollary 4.1. They show a modest visual effect of the choice of approximate initial condition, and illustrate significant improvement in the stability and accuracy of reconstructions in the presence of noise when the linear system used to produce the weights is regularized. Lastly, they provide confirmation of the remedy that a signed measure provides to approximate solutions defined by first order regularization when $\nu \geq 4$.

5.1. Numerical implementation

Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a partition of $[0, T]$ with $t_i = i\Delta t$ and $\Delta t = T/N$. We assume a solution of the form

$$u^N(t) := \sum_{i=0}^{N} u^N_i \phi^N_i(t),$$

(68)

where the $\phi^N_i$ are the usual “hat” basis functions for the space of continuous piecewise linear functions on $[0, T]$ (see e.g. [8]). To facilitate the use of data on an extended interval $[0, T + \bar{\alpha}]$, define $\bar{R} = \lfloor N/2 \rfloor + 1$ and let $\alpha = \alpha(r) = (r+1)\Delta t$ for $r \in \{1, 2, \ldots, \bar{R} - 1\}$. This allows us obtain additional grid points $t_i = i\Delta t$ for $i = N + 1, \ldots, N + \bar{R}$.

The discrete admissible measure $w_{\alpha(r)}$ is that defined in (63) with $L = r$ and $\tau_{\ell} = (\ell+1)/(r+1)$, $\ell = 0, \ldots, r$. An approach to choose the $\beta$-dependent ($\beta \geq 0$) signed weights, $w_{\ell}(\beta)$, $\ell = 0, \ldots, r$, is to first suppose that the polynomial in (18) is given simply by

$$p_{\nu}(t) = (t + \bar{m})^{\nu + 1},$$

(69)

for some $\bar{m} > 0$. Then using a binomial coefficient expansion, condition (i) in Definition 2.1 imposes the requirement, for $\beta \geq 0$,

$$\sum_{\ell=0}^{r} w_{\ell}(\beta) \binom{\ell + 1}{r + 1}^{\nu + 1 + j} = \frac{(\nu + 1)!}{(\nu + 1 - j)!} \bar{m}^{\nu + 1 - j},$$

(70)

for $j = 0, \ldots, \nu + 1$. A (least-squares) solution $(w_0(\beta), \ldots, w_r(\beta))$ of the $(\nu + 2)$-dimensional (ill-conditioned) system in (70) is obtained in a stable manner using
a small Tikhonov regularization parameter \( \beta > 0 \); in this case, one obtains new coefficients \( \tilde{c}_{\nu+1}, \ldots, \tilde{c}_{2\nu+2} \) and it must be checked that the roots of the resulting polynomial \( \tilde{p}_\nu(t) = \sum_{j=0}^{\nu+1} \tilde{c}_{\nu+j+1}/j! t^j \) have negative real part. Note that Lemma 4.1 guarantees that the root condition is satisfied for \( \beta \) sufficiently small in the special case that \( L = \nu + 1 \) and the nodes are chosen to ensure that for all \( \ell = 0, \ldots, \nu + 1 \), the nodes satisfy \( \tau_i \alpha(r) = t_i \) for some \( i \in 0, \ldots, N + \tilde{R} \).

With weights \( w_\nu(\beta) \) chosen, we seek a vector of \( \alpha(r) \)-dependent coefficients \( (u_0^N, \ldots, u_N^N) \) that satisfy the \((N+1)\)-dimensional linear system of collocation equations

\[
a_{\alpha(r)} u_i^N(t_i) + b_{\alpha(r)}(u_i^N)'(t_i) + A_{\alpha(r)} u_i^N(t_i) = T_{\alpha(r)} f^i(t_i),
\]

where \( i = 0, \ldots, N \), and as in [9] (assuming \( \eta_\alpha \) in (71) is no longer normalized),

\[
\kappa_\ell := \int_0^{t_{\ell+1}} k(s)ds \quad \lambda_\ell := \int_0^{t_{\ell+1}} k(s)(t_{\ell+1} - s)ds, \quad \ell = 0, \ldots, r,
\]

and suppressing the dependence on \( \beta \),

\[
a_{\alpha(r)} := \sum_{\ell=0}^r \kappa_\ell \lambda_\ell w_\ell \quad b_{\alpha(r)} := \sum_{\ell=0}^r \lambda_\ell^2 w_\ell \quad c_{\alpha(r)} := \sum_{\ell=0}^r \kappa_\ell^2 w_\ell,
\]

\[
T_{\alpha(r)} f^i(t_i) := \sum_{\ell=0}^r f^i(t_{\ell+1}) \lambda_\ell w_\ell \quad \tilde{g}_{\alpha(r)} := \sum_{\ell=0}^r f^i(t_{\ell+1}) \kappa_\ell w_\ell,
\]

and

\[
A_{\alpha(r)} u_i^N(t_i) := \sum_{j=0}^i u_j^N \sum_{\ell=0}^r \lambda_\ell w_\ell \int_0^{t_i} k(t_{i+\ell+1} - s) \phi_j(s)ds.
\]

Now set \( u_0^N = u^N(t_0) \) to be the initial value given either by \( \bar{u}(0) \), the first-order approximate (51), or the zeroth-order approximate (14), and solve the system in (71) sequentially by setting

\[
(u_i^N)' = \frac{1}{b_{\alpha(r)}} (T_{\alpha(r)} f^i(t_i) - a_{\alpha(r)} u_i^N - A_{\alpha(r)} u_i^N(t_i))
\]

\[
u_{i+1}^N = u_i^N + (u_i^N)' \Delta t,
\]

for \( i = 0, \ldots, N - 1 \), to obtain the \( \alpha(r) \)-dependent approximation \( u^N(t) \) on \([0, T]\) given by the formula in (68).

5.2. Examples

The examples below illustrate numerical approximations for values of \( \nu = 3, 4 \) and 5. To obtain the signed weights, we choose \( \bar{m} = \nu \) in (69) and obtain the least squares solution to (70) with and without the use of Tikhonov regularization with specified parameter \( \beta \). Reconstructions with the first order method in [9] are performed using the uniform positive weights \( w_\ell = 1 \) for all \( \ell = 0, \ldots, r \). We refer to the three types of reconstructions as “stable signed weight,” “signed weight,” and “uniform positive weight,” respectively. All methods are given the exact initial value except as specified within Example 5.3 below.
To simulate noisy data, we construct the exact data vector $\mathbf{f} \in \mathbb{R}^{N+R+1}$, where $(\mathbf{f})_{j+1} = f(t_j)$ for $j = 0, \ldots, N + R$, and add to it a random vector $\mathbf{n}$ of Gaussian noise in $(0, 1)$ scaled by the factor $\sigma_n \|\mathbf{f}\|_\infty$. The absolute error is $\delta := \sigma_n \|\mathbf{f}\|_\infty \|\mathbf{n}\|_\infty$, and $\mathbf{u}_{\mathbf{N},\alpha(r)} \in \mathbb{R}^{N+1}$ denotes the approximate solution vector constructed using the first $(N + r + 2)$ entries of the noisy data vector. Similarly, $\mathbf{\bar{u}}_N \in \mathbb{R}^{N+1}$ denotes a vector whose entries are the exact solution to (1) evaluated at the first $(N + 1)$ grid points.

We set $N = 600$. Unless otherwise stated, dashed lines represent the true solution, solid lines denote reconstructions, and the value of $\alpha(r)$ is chosen optimally to minimize the relative solution error in the Euclidean norm, $\|\mathbf{u}_{\mathbf{N},\alpha(r)} - \mathbf{\bar{u}}_N\|_2 / \|\mathbf{\bar{u}}_N\|_2$.

**Example 5.1.** We first demonstrate the stability, convergence, and rate results for the improved method with stable signed weight reconstructions ($\beta = 0.0002$) of $\bar{u}(t) = \cos(4t)$ on $[0, 3]$ in the case $\nu = 3$. Simulated noisy data is created with the values $\sigma_n = 0, 0.000925, 0.0019, 0.0037, 0.0074,$ and $0.0148$.

Stability and convergence of the method are illustrated in Figure 1 and by Table 1. Table 2 shows that the asymptotic behavior of the solution error coincides precisely with that proposed in Corollary 4.1 (for $\zeta = 1$) when the parameter is chosen according to the theoretical value $\alpha = \delta^{1/5}$.

![Figure 1: Stable signed weight reconstructions from data with 0%, 0.25% 0.5%, 1%, 2% and 4% relative noise and $\nu = 3$ (from left to right).](image)

**Example 5.2.** In Figure 2, we compare the performance of the three methods to approximate $\bar{u}(t) = \cos(4t)$ on $[0, 12]$ in the case $\nu = 4$. The uniform positive weight reconstructions coincide with those displayed in [9] and illustrate that method’s instability as the root condition in Definition 2.1 (condition (A3) in that reference) is no longer satisfied in the case of positive weights. Although there is a scheme in [5] to construct a positive measure for zeroth order local regularization in the case $\nu = 4$, no such construction is available for the first order method.

![Figure 2](image)
\[ \delta \alpha(\tau) = \left\lfloor \frac{\delta^{1/5}}{\Delta t} \right\rfloor \Delta t \]
\[ \| u_{N,\alpha(\tau)}^\delta - \bar{u}_N \|_2 / \| \bar{u}_N \|_2 \]

<table>
<thead>
<tr>
<th>( \delta )</th>
<th>( \delta / | f |_{\infty} )</th>
<th>( \alpha(\tau) )</th>
<th>( | u_{N,\alpha(\tau)}^\delta - \bar{u}_N |_2 / | \bar{u}_N |_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d )</td>
<td>0.04043</td>
<td>0.85</td>
<td>0.3103</td>
</tr>
<tr>
<td>( d/2 )</td>
<td>0.02021</td>
<td>0.76</td>
<td>0.2214</td>
</tr>
<tr>
<td>( d/4 )</td>
<td>0.01011</td>
<td>0.67</td>
<td>0.1580</td>
</tr>
<tr>
<td>( d/8 )</td>
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<td>0.60</td>
<td>0.1134</td>
</tr>
<tr>
<td>( d/16 )</td>
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<td>0.54</td>
<td>0.08235</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0.195</td>
<td>0.02825</td>
</tr>
</tbody>
</table>

Table 1: Stability and convergence of stable signed weight reconstructions, \( d = 0.02370 \) and optimal \( \alpha(\tau) \).

\[ \delta \alpha(\tau) = \left\lfloor \frac{\delta^{1/5}}{\Delta t} \right\rfloor \Delta t \]
\[ \| u_{N,\alpha(\tau)}^\delta - \bar{u}_N \|_2 / \delta^{2/5} \]

<table>
<thead>
<tr>
<th>( \delta )</th>
<th>( \alpha(\tau) )</th>
<th>( | u_{N,\alpha(\tau)}^\delta - \bar{u}_N |_2 / \delta^{2/5} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d )</td>
<td>0.47</td>
<td>200.0</td>
</tr>
<tr>
<td>( d/2 )</td>
<td>0.41</td>
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</tr>
<tr>
<td>( d/4 )</td>
<td>0.355</td>
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</tr>
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<td>( d/8 )</td>
<td>0.31</td>
<td>196.8</td>
</tr>
<tr>
<td>( d/16 )</td>
<td>0.27</td>
<td>200.2</td>
</tr>
</tbody>
</table>

Table 2: Rate of convergence of stable signed weight reconstructions with \( \alpha \) chosen according to Corollary 4.1, \( \zeta = 1 \), \( \nu = 3 \), and \( d = 0.02370 \).

The signed weight reconstructions appear to approximate \( \bar{u} \) with nearly the same degree of accuracy when noise free data is used. However, the stable signed weight reconstruction \( (\beta = 0.0002) \) is somewhat less accurate with a relative solution error of 29.1% versus 19.5% with no stabilization.

**Example 5.3.** We now illustrate in Figure 3 the effect of Tikhonov regularization of the associated ill-conditioned system (70) to obtain signed weights in the case \( \nu = 4 \). The result is reduced variability in the magnitude of the weights and superior stability that leads to an accurate reconstruction of \( \bar{u}(t) = \cos(4t) \) on \([0,12] \) with 2% relative noise in the data; the relative solution error is 30.3% with a stable signed weight versus 99.6% when the weight is obtained with no regularization.

We also illustrate with this same example the effect of the choice of initial value on the stable weight reconstruction in the presence of noise. The initial values used are \( \bar{u}(0) \) (solid black), the first-order approximate (dashed-dotted line), and zeroth-order approximate (dotted line). All three reconstructions appear to perform well regardless of the given initial value. In Figure 4, we zoom in on the interval \([0,2] \) and plot every other reconstruction value to better discern behavior near \( t = 0 \). Notice that the reconstruction using the zeroth-order approximate
Figure 2: Noise free reconstructions with $\nu = 4$. Top row: Uniform positive weights with $\alpha = 0.8$ and $\alpha = 1.6$, respectively. Bottom row: Stable signed weight with $\alpha = 0.92$ and signed weight with $\alpha = 0.94$.

initial condition aligns (after a small initial interval) almost identically with the numerical solution obtained using the exact initial value.

Example 5.4. Last, we illustrate how a signed measure remedies the stability issue encountered with positive measures used in [9] allowing for accurate reconstructions in the case $\nu = 5$. Figure 5 shows examples in which noise-free data on $[0, 4.145]$ and $0.1\%$ noisy data on $[0, 4.5]$ are used to obtain stable signed weight approximates of $\bar{u}(t) = t^4 - 3t^3 + 2t$ on $[0, 3]$. A smaller parameter $\beta = 0.00001$ is used to ensure that condition (i) in Definition 2.1 is satisfied.

As the value of $\nu$ increases so does the ill-posedness of the problem; solutions are increasingly sensitive to noise and it is expected that additional data would be needed to achieve accurate reconstructions. It is worth mentioning that an advantage of the method lies in the ability to extend the domain for data past $[0, 3]$. If additional data is unavailable, one could instead obtain an approximate solution on a shortened interval $[0, 3 - \bar{\alpha} + \alpha]$ (this approach was taken in [1]) else be faced with reconstructions of decreased quality for large values of $\nu$.

6. Higher order local regularization

The focus of this paper is a method of first order local regularization for a class of ill-posed Volterra equations. However, in view of equation (7) in Section 1, there is a natural extension of the ideas herein to higher order reg-
Figure 3: Signed weight and stable signed weight reconstructions from data with 2% relative noise and $\nu = 4$, optimal $\alpha = 1.54$ and $\alpha = 0.94$, respectively (from left to right).

Figure 4: Stable signed weight reconstructions from data with 2% relative noise and $\nu = 4$ near $t = 0$; exact (solid), first order approximate (dashed-dotted), and zeroth order (dotted) approximate initial conditions with optimal values of $\alpha = 0.94$, $\alpha = 0.98$ and $\alpha = 0.94$, respectively.

We briefly describe the latter along the lines of the generalized theory of zeroth order regularization in [1]. A theoretical framework for this development is apparent in the part of the proof of Theorem 3.1 which occurs after a translation to zero initial values. As higher order methods generally require nonzero initial values, such conditions have to be handled separately, much as is done in Section 3.1.

To proceed with an abstract generalization, let $X$ denote a Banach space (which corresponds to $L^p(I)$ or $C(I)$ in this paper) and take as the original problem that of finding $\bar{u} \in X$ satisfying $Au = f$, given $f \in X$ and $A \in \mathcal{L}(X)$. Given perturbed data $f^\delta \in X$ and $\alpha \in \Lambda$, approximations $u^\delta_\alpha$ of $\bar{u}$ are found in the space $X_\alpha$ which, in a slight generalization, is allowed to depend on $\alpha$. (Recall that earlier the approximation space – either $L^p(I_0)$ or $C(I_0)$ – was assumed to be $\alpha$-independent.)
The basic assumptions are as follows:

A1. Let \([X_\alpha, r_\alpha]\) denote the pairing of a Banach space \((X_\alpha, \|\cdot\|_\alpha)\) and a well-defined linear operator \(r_\alpha : X \mapsto X_\alpha\) which serves to facilitate movement between the two spaces.

A2. Let the “data sampling” operator \(T_\alpha \in \mathcal{L}(X, X_\alpha)\) satisfy \(\|T_\alpha g\|_\alpha \leq M\|g\|\), \(g \in X\), for \(M > 0\) independent of \(\alpha \in \Lambda\).

A3. The operator \(T_\alpha A\) may be written \(T_\alpha A = D_\alpha + A_\alpha r_\alpha\), for \(D_\alpha \in \mathcal{L}(X, X_\alpha)\) and \(A_\alpha \in \mathcal{L}(X_\alpha)\), where for some \(a_\alpha \neq 0\) and for \(L_\alpha : \text{dom}(L_\alpha) \subseteq X_\alpha \mapsto X_\alpha\) a linear operator, the following is true.

(i) The operator \((a_\alpha L_\alpha + A_\alpha) : \text{dom}(L_\alpha) \subseteq X_\alpha \mapsto X_\alpha\) has a bounded inverse for all \(\alpha \in \Lambda\) sufficiently small, with operator norm satisfying

\[
\|(a_\alpha L_\alpha + A_\alpha)^{-1}\|_{\mathcal{L}(X_\alpha)} \leq \frac{1}{c(\alpha)}
\]

for some \(c(\alpha) > 0\).

(ii) The operator \(D_\alpha\) is approximated by \(a_\alpha L_\alpha r_\alpha\) on a nonempty subspace \(S \subseteq \{u \in X, \ r u \in \text{dom}(L_\alpha), \ \alpha \in \Lambda\}\) of \(X\) in the sense that, for each \(u \in S\),

\[
\|(D_\alpha - a_\alpha L_\alpha r_\alpha)u\|_\alpha = o(c(\alpha)) \quad \text{as} \ \alpha \to 0^+.
\]  

In the following theorem, we confirm that the solution \(u_\alpha^\delta\) of the general local regularization equation (15) converges to \(\bar{u}\) in an appropriate sense. The verification is similar to part of the proof of Theorem 3.1.

**Theorem 6.1.** Assume that A1-A3 hold for all \(\alpha \in \Lambda\). Then for each \(f^\delta \in X\) and \(\alpha \in \Lambda\), equation (15) has a unique solution \(u_\alpha^\delta \in X_\alpha\) which depends continuously on \(f^\delta \in X\). Further, given \(\{f^\delta\}_{\delta > 0}\) satisfying \(\|f^\delta - f\| \leq \delta\) and a selection \(\alpha = \alpha(\delta) \in \Lambda\) such that

\[
\alpha(\delta) \to 0 \quad \text{and} \quad \frac{\delta}{c(\alpha(\delta))} \to 0, \quad \text{as} \ \delta \to 0,
\]

Figure 5: Stable signed weight reconstructions with \(\nu = 5\) from data with 0% and 0.1% noise and optimal \(\alpha = 1.145\) and \(\alpha = 1.5\), respectively (from left to right).
then if \( \bar{u} \in S \),
\[
\| u_{\alpha(\delta)}^\delta - r_{\alpha(\delta)} \bar{u} \|_{\alpha(\delta)} \to 0, \quad \text{as } \delta \to 0.
\]

In addition, if \( \bar{u} \in S \) is such that, instead of (72), we have
\[
\|(D_\alpha - a_\alpha L_\alpha r_\alpha) \bar{u}\|_\alpha = \bar{w}(\alpha)c(\alpha) \to 0, \quad \text{as } \alpha \to 0^+
\]
then
\[
\| u_{\alpha(\delta)}^\delta - r_{\alpha(\delta)} \bar{u} \|_{\alpha(\delta)} = O \left( \frac{\delta}{c(\alpha(\delta)) + \bar{w}(\alpha(\delta))} \right) \quad \text{as } \delta \to 0.
\]

7. Conclusion

It states in [9] that the construction of a stable local (sequential) regularization method for \( \nu \)-smoothing Volterra equations with \( \nu \geq 5 \) is “still an open problem”. For zeroth order local regularization methods, this problem is solved in [7] and further generalized in [1]. The purpose of this paper is to overcome the same limitation in the case of first order local regularization. Indeed, using a derivation based on a well-defined class of signed measures, we demonstrate that stability and convergence of first order local regularization methods are obtained for all \( \nu = 1, 2, \ldots \).

It is shown that the needed measures are easy to construct in the case of either \( L^p \) or continuous data and that, while the method requires an estimate \( u_0 \) of \( \bar{u}(0) \), two theoretically-sound estimation approaches (one due to [9]) are possible. In numerical testing of the fast, sequential algorithm, we confirm the expected convergence rate and illustrate the effectiveness of the method. Additionally, we offer a way to generalize the theoretical findings in this paper to higher order local regularization methods.

Theoretical findings also demonstrate that first order and zeroth order local regularization (see [1, 2, 7]) share the same convergence rate when the methods are applied in the case of solutions \( \bar{u} \) enjoying additional regularity properties (compare, for example, Corollary 4.1 above with Theorem 3.2 of [1]) in the case of \( \bar{u} \in C^2(I) \) with \( \bar{u}(0) = \bar{u}'(0) = 0 \). However, numerical tests indicate that first order methods offer smoother visual reconstructions of such solutions when compared to other sequential methods [9]. Surely, one cannot expect reasonable reconstructions from first order methods if \( \bar{u} \) is only in \( L^p(I) \) (or \( C(I) \)) so in this case, zeroth order regularization may be the more appropriate approach.

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References


