\[ \lim_{x \to 0} \frac{\tan x}{x} \]

**Note**

\[ \lim_{x \to 0} \tan x = 0 \]
\[ \lim_{x \to 0} x = 0 \]

So we have \( \frac{0}{0} \), indeterminate.

\[ = \lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\sin x}{(\cos x) \cdot x} = \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x} \]

\[ = \left( \lim_{x \to 0} \frac{\sin x}{x} \right) \left( \lim_{x \to 0} \frac{1}{\cos x} \right) = \left( 1 \right) \left( \frac{\lim_{x \to 0} 1}{\lim_{x \to 0} \cos x} \right) \]

\[ = \left( 1 \right) \left( \frac{1}{\cos 0} \right) = \left( 1 \right) \left( \frac{1}{1} \right) = 1 \]

\[ \lim_{x \to 0} \left( e^{-x} \sin(\pi x) \right) = \left( \lim_{x \to 0} e^{-x} \right) \left( \lim_{x \to 0} \sin(\pi x) \right) \]

\[ = \left( 1 \right) \left( \sin \left( \lim_{x \to 0} \pi x \right) \right) = \left( 1 \right) \left( \sin(0) \right) = \left( 1 \right) \left( 0 \right) = 0 \]

Try following at home:

1) \[ \lim_{x \to -1} \frac{(2x-3)(x+1)}{x+1} \]

2) \[ \lim_{x \to 0} \frac{\sqrt{x+5} - \sqrt{5}}{x} \]

3) \[ \lim_{x \to 0} \frac{\sin(x)(1-\cos x)}{2x^2} \]

4) \[ \lim_{x \to 1^-} \frac{x^2 + 2x - 3}{|x-1|} \]
\[ f(x) = \frac{x+1}{x^2 - 2x - 3} \]

Find:

a) \( \lim_{x \to 3^-} f(x) \)

b) \( \lim_{x \to 3^+} f(x) \)

c) \( \lim_{x \to 3} f(x) \)

We can write:

\[ f(x) = \frac{x+1}{(x+1)(x-3)} = \frac{1}{x-3} \]

\[ \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \frac{1}{x-3} = -\infty \]

\[ \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} \frac{1}{x-3} = \infty \]

\[ \lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{1}{x-3} = \text{DNE} \]

\[ \lim_{x \to -1} f(x) = \lim_{x \to -1} \frac{x+1}{(x+1)(x-3)} = \lim_{x \to -1} \frac{1}{x-3} = \frac{-1}{4} \]
Squeeze theorem

If \( f(x) \leq g(x) \leq h(x) \) when \( x \) is near \( c \) (except possibly at \( c \)) and

\[
\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L
\]

then

---

**Example**

If \( 2x - 1 \leq f(x) \leq x^2 - 2x + 3 \) for \( x > 0 \), find \( \lim_{x \to 2} f(x) \).

\[
\lim_{x \to 2} 2x - 1 = 3
\]

\[
\lim_{x \to 2} x^2 - 2x + 3 = 2^2 - 2 \cdot 2 + 3 = 3
\]

From squeeze theorem

\[
\lim_{x \to 2} f(x) = 3
\]
2.4 Precise definition of limits

We learnt (17) limit laws, but do we derive those rules.

\[ \lim_{x \to 0} \frac{\sin x}{x} = 1 \]  
how do we know it is exactly 1?

**Definition**

Let \( f \) be a function defined on some open interval that contains the number \( a \), except possibly at \( a \) itself.

Then we say that the limit of \( f(x) \) as \( x \) approaches \( a \) is \( L \), and we write

\[ \lim_{x \to a} f(x) = L \]

For every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that

if \( 0 < |x - a| < \delta \) then \( |f(x) - L| < \varepsilon \)

\[ \lim_{x \to 2} x^2 = 4 \]

How small the \( \varepsilon \) is, we should be able to find an open interval which includes 2 in it.
\[ \lim_{x \to 2} f(x) \]

\[ \lim_{x \to 2} f(x) = 4 \quad ? \]

Every \( \varepsilon > 0 \) will not lead to an open interval that contains 2.

i.e. \( \lim_{x \to 2} f(x) \neq 4 \).

**Example:**

Use the graph to find \( \delta > 0 \) such that for all \( x \),

\[ 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon \]

when \( f(x) = \sqrt{x - 2} \) , \( x_0 = 4 \) , \( L = \sqrt{2} \) , \( \varepsilon = \frac{1}{4} \)

\[ \sqrt{2} - \frac{1}{4} = \sqrt{x - 2} \]

\[ (\sqrt{2} - \frac{1}{4})^2 = x - 2 \]

\[ x = (\sqrt{2} - \frac{1}{4})^2 + 2 \approx 3.3554 \]
To find $b$

\[
\sqrt{2} + \frac{1}{4} = \sqrt{x - 2}
\]

\[
x = (\sqrt{2} + \frac{1}{4})^2 + 2 \approx 4.7696
\]

So $b = 0.6446$