\[ \lim_{x \to 3^+} \frac{x}{x^2 - 9} \to 0^+ \]
\[ \lim_{x \to 3^-} \ln \left[ \frac{x^2}{3-x} \right] \to -\infty \]

2.5 **Continuity at a point**

If a function \( f(x) \) is continuous at point \( 'c' \) then

\[ \lim_{x \to c} f(x) = f(c) \]

i.e. The left limit and right limit at \( 'c' \) should be equal to function value at \( 'c' \)

Above functions are not continuous at \( x = c \). But they are removable discontinuities.
Jump discontinuity

Above, is not continuous at $x = c$. These are non-removable discontinuities.

One-sided limits

Right limit $\neq f(c)$

So $f(x)$ is not right continuous at $x = c$.

Since $\lim_{x \to c^-} f(x) = f(c)$

$f(x)$ is left continuous at $x = c$. 
\[ f(x) = \begin{cases} 
1 & x < -1 \\
3 & x = -1 \\
x + 2 & -1 < x \leq 0 \\
e^x & 0 < x 
\end{cases} \]

At \( x = -1 \):

\[
\lim_{x \to -1} f(x) = 1 \neq f(-1)
\]

So \( f(x) \) is not continuous at \( x = -1 \). \( x = -1 \) is a removable discontinuity.

At \( x = 0 \):

\[
\lim_{x \to 0} f(x) = \text{DNE}
\]

So \( f(x) \) is not continuous at \( x = 0 \). \( x = 0 \) is a non-removable discontinuity.
Continuity on a closed interval.

A function \( f \) is continuous on a closed interval \([a, b]\) if it is continuous on the open interval \((a, b)\) and

\[
\lim_{{x \to a^+}} f(x) = \lim_{{x \to b^-}} f(x) = \]

Continuity at point \( c \) is destroyed when:
- function is not defined at \( x = c \)
- limit does not exist at \( x = c \)
- limit is not equal to function value at \( x = c \).

8. Discuss the continuity of \( f(x) = \frac{1}{x} \).

\[
\lim_{{x \to c}} f(x) = \lim_{{x \to c}} \frac{1}{x} = \frac{1}{c} = f(c) \quad , \quad c \neq 0
\]

\[
\lim_{{x \to c}} f(x) = f(c) \quad , \quad c \neq 0
\]

So \( f(x) \) is continuous everywhere except possibly for \( x = 0 \).
Since \( \lim_{x \to 0^+} \frac{1}{x} = \infty \) and \( \lim_{x \to 0^-} \frac{1}{x} = -\infty \)

\[ \lim_{x \to 0} f(x) = \text{DNE} \]

i.e. \( f(x) \) is not continuous at \( x=0 \).

**Example:** Discuss the continuity of \( f(x) = \frac{x^2-1}{x-1} \)

\[ \lim_{x \to c} f(x) = \lim_{x \to c} \frac{x^2-1}{x-1} = \frac{c^2-1}{c-1} = f(c) \quad c \neq 1 \]

\[ \lim_{x \to c} f(x) = f(c) \quad c \neq 1 \]

So \( f(x) \) is continuous everywhere except possibly at \( x=1 \).

At \( x=1 \)

\[ \lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x \to 1} \frac{(x+1)(x-1)}{(x-1)} = 2 \]

\[ f(1) = \frac{1^2-1}{1-1} = \text{und} \]

\( f(x) \) is not continuous at \( x=1 \). But \( x=1 \) is a removable discontinuity.
9. Find the continuity of \( f(x) = \sqrt{1 - x^2} \).

Note: Domain of \( f(x) \) is \([-1, 1]\)

When \( c \in (-1, 1) \)

\[
\lim_{x \to c} f(x) = \lim_{x \to c} \sqrt{1 - x^2} = \sqrt{\lim_{x \to c} (1 - x^2)} = \sqrt{1 - c^2} = f(c)
\]

\[
\lim_{x \to c} f(x) = f(c)
\]

\( f(x) \) is continuous on \((-1, 1)\).

At \( x = -1 \)

\[
\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} \sqrt{1 - x^2} = \sqrt{\lim_{x \to -1^+} 1 - x^2} = \sqrt{1 - (-1)^2} = 0
\]

\[
= f(c) \quad f(-1)
\]

\( f(x) \) is right continuous at \( x = -1 \)

At \( x = 1 \)

\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \sqrt{1 - x^2} = \sqrt{\lim_{x \to 1^-} (1 - x^2)} = \sqrt{1 - (1)^2} = 0
\]

\[
= f(1)
\]

\( f(x) \) is left continuous at \( x = 1 \)

\( f(x) \) is continuous on closed interval \([-1, 1]\)
Vertical asymptotes

The line $x = c$ is called a vertical asymptote of the curve $f(x)$ if

$$\lim_{x \to c^-} f(x) = \pm \infty \quad \text{and/or} \quad \lim_{x \to c^+} f(x) = \pm \infty$$

**Q** Find vertical asymptote of $f(x) = \frac{x^2 + 1}{x^2 - 1}$

Since denominator becomes zero at $x = 1$ and $x = -1$ we suspect vertical asymptotes at $x = 1$ and $x = -1$.

**At $x = 1$**

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^2 + 1}{x^2 - 1} = \pm \infty$$

So $x = 1$ is a vertical asymptote.

**At $x = -1$**

$$\lim_{x \to -1} f(x) = \lim_{x \to -1} \frac{x^2 + 1}{x^2 - 1} = \pm \infty$$

So $x = -1$ is a vertical asymptote.
8. Find vertical asymptote \( f(x) = \frac{x}{\sin x} \)

Denominator becomes zero at \( x = 0, \pm \pi, \pm 2\pi, \ldots, \pm n\pi \)

At \( x = 0 \)

\[
\lim_{x \to 0} \frac{x}{\sin x} = \lim_{x \to 0} \frac{1}{\frac{\sin x}{x}} = \frac{1}{1} = 1
\]

So \( x = 0 \) is not a vertical asymptote. It is a removable discontinuity.

At \( x = \pm n\pi, n \neq 0 \)

\[
\lim_{x \to \pm n\pi} \frac{x}{\sin x} = \lim_{x \to \pm n\pi} \frac{\pm n\pi}{\sin \pm n\pi} = \pm \infty
\]

So \( x = \pm n\pi, n \neq 0 \) are vertical asymptotes.

**Horizontal Asymptotes**

The line \( y = L \) is called a horizontal asymptote of the curve \( y = f(x) \) if either

\[
\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L
\]
\[ \lim_{x \to \infty} \frac{x^2 - 1}{x^2 + 1} = \lim_{x \to \infty} \frac{(x^2 - 1)/x^2}{(x^2 + 1)/x^2} = \lim_{x \to \infty} \frac{1 - \frac{1}{x^2}}{1 + \frac{1}{x^2}} = 0 \]

So \( y = 0 \) is a horizontal asymptote.

\[ \lim_{x \to -\infty} \frac{x^2 - 1}{x^2 + 1} = \lim_{x \to -\infty} \frac{(x^2 - 1)/x^2}{(x^2 + 1)/x^2} = \lim_{x \to -\infty} \frac{1 - \frac{1}{x^2}}{1 + \frac{1}{x^2}} = 0 \]

So \( y = 0 \) is a horizontal asymptote.

\[ \text{Eg find horizontal asymptote of } f(x) = \frac{3x^2 - \sqrt{x-2}}{5x^2 + 4x + 1} \]

\[ \lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{3x^2 - \sqrt{x-2}}{5x^2 + 4x + 1} = \lim_{x \to \infty} \frac{(3x^2 - \sqrt{x-2})/x^2}{(5x^2 + 4x + 1)/x^2} \]

\[ = \lim_{x \to \infty} \frac{3 - \sqrt{\frac{x-2}{x^4}}}{5 + \frac{4}{x} + \frac{1}{x^2}} = \lim_{x \to \infty} \frac{3 - \sqrt{\frac{1}{x^3} - \frac{3}{x^4}}}{5 + \frac{4}{x} + \frac{1}{x^2}} = \frac{3}{5} \]

So \( y = 3/5 \) is horizontal asymptote.

Note \( \lim_{x \to -\infty} f(x) \) will produce the same horizontal asymptote.