3.10 Linear approximation and differentials

Here we obtain a linear approximation for a complicated function \( y = f(x) \).

Equation of tangent line:

\[
y - y_0 = m(x - x_0).
\]

\[
y - f(a) = f'(a)(x-a)
\]

\[
y = f(a) + f'(a)(x-a)
\]

\[
L(x) = f(a) + f'(a)(x-a)
\]

linear approx of \( f(x) \) at \( x=a \)
(a) Find the linear approx of \( y = \sqrt{x+3} \) at \( x = 1 \).

Slope of tangent
\[
y'(x) = \frac{1}{2} (x+3)^{-1/2} = \frac{1}{2 \sqrt{x+3}}
\]

Slope at \( x = 1 \)
\[
y'(1) = \frac{1}{2 \sqrt{4+3}} = \frac{1}{2 \cdot 4} = \frac{1}{4}
\]

Point on tangent
\[
y(1) = \sqrt{1+3} = \sqrt{4} = 2
\]

Eqn of tangent
\[
y - 2 = \frac{1}{4} (x-1)
\]

\[
y - 2 = \frac{x}{4} - \frac{1}{4}
\]

\[
y = \frac{x}{4} - \frac{1}{4} + \frac{2}{4}
\]

\[
y = \frac{x}{4} + \frac{1}{4}
\]

\[
L(x) = \frac{x}{4} + \frac{1}{4}
\]

(b) Approximate \( \sqrt{3.98} \)

View line \( \sqrt{3.98} = \sqrt{0.98+3} \). To approximate, we will use linear approx at \( x = 0.98 \)

\[
L(0.98) = \frac{0.98}{4} + \frac{1}{4} \approx 1.995
\]

Note: Actual value is 1.999
Differentials

\[ y = f(x) \]

\[ \Delta y \quad \text{dy - differential} \]
\[ \Delta y \quad \text{Actual change in } y \text{ due to } \Delta x \text{ change in } x \]

Differential

\[ dy = f'(x_0) \Delta x \]

If \( \Delta x \) is very small, \( \Delta y \approx dy \)

\[ f'(x) = x^3 + x^2 - 2x + 1 \]

\[ f'(x) = 3x^2 + 2x - 2 \]
\[ f'(2) = 3 \cdot 2^2 + 2 \cdot 2 - 3 = 14. \]

So differential.

\[ dy = f'(x_0) \Delta x \]
\[ dy = 14 \cdot \Delta x. \]

(6) Approximate the change in \( y \), if \( x \) changes from \( x = 2 \) to \( x = 2.01 \).

\[ \Delta x = 2.01 - 2 = 0.01. \]

This is a very small change.

Since \( \Delta x \) is small, \( \Delta y \approx dy \)

but \[ dy = 14 \cdot (0.01) = 0.14 \]

So \[ \Delta y \approx 0.14. \]
Eq. The radius of a sphere was measured and found, 21 cm with a possible error in measurement of at most \( \pm 0.05 \text{ cm} \).

What is the max error in using this value of radius to compute the volume of the sphere. Also find the relative error.

Eq. between \( V \) and \( r \) is

\[
V = \frac{4}{3} \pi r^3.
\]

Let us find differential.

\[
dV = \left[ \text{derivative of volume at } r=21 \right] \cdot \Delta r
\]

\[
= \left[ \frac{4\pi}{3} r^2 \text{ at } r=21 \right] \cdot \Delta r
\]

\[
= \left[ 4\pi (21)^2 \right] \Delta r.
\]

Since change in radius is at most \( \Delta r = \pm 0.05 \)

\[
dV = 4\pi (21)^2 (\pm 0.05)
\]

\[
= \pm 277 \text{ cm}^3.
\]

Max error in volume is 277 cm\(^3\).

Relative error: \[
\frac{\Delta V}{V} \text{ actual change } \quad \frac{dV}{V} \text{ nominal volume}
\]

Since \( \Delta r \) is very small, \( \Delta V \approx dV \)

So Relative error \( = \frac{\Delta V}{V} \approx \frac{dV}{V} = \frac{277}{\frac{4\pi (21)^3}{3}} = 0.007 \)

\( 0.7\% \).