A Well-Known Result and its Dependency on the Axiom of Choice

Santiago Salazar

Department of Mathematics

Abstract

The combined results of Kurt Gödel and Paul Cohen show that the the Axiom of Choice (AC) is independent from the Zermelo-Fraenkel (ZF) axioms of set theory [2]. For this reason, the AC is appended to the ZF axioms to form what is denoted as the ZFC (ZF + AC). This poster illustrates the intrinsic dependency of a well-known mathematical result on the AC.

Background

The ZF axioms are as follows:

The Axiom of Extension

Two sets are equal if and only if they have the same elements.

The Axiom of Specification

To every set A and every condition $S(x)$ there corresponds a set B whose elements are exactly those elements x of A for which $S(x)$ holds.

The Axiom of Pairing

For any two sets there exists a set that they both belong to.

The Axiom of Union

For every collection of sets there exists a set that contains all the elements that belong to at least one set of the given collection.

The Axiom of Power

For each set there exists a collection of sets that contains among its elements all the subsets of the given set.

The Axiom of Infinity

There exists a set containing 0 (that is, the empty set $\emptyset$) and containing the successor of each of its elements.

Outcomes of the ZF System

Surprisingly, from this collection of axioms, the existence of all the indispensable mathematical tools can be proved. For instance, the construction of the natural numbers together with Peano's axioms as well as the rational numbers and the real numbers can be obtained just from the ZF axioms alone [4].

The Axiom of Choice

It turns out that the AC is equivalent to several other mathematical results. Below are some just some of them [2] [4].

The Axiom of Choice

The Cartesian product of a non-empty family of non-empty sets is non-empty.

Zorn's Lemma

If X is a partially ordered set such that every chain in X has an upper bound, then X contains a maximal element.

Well Ordering Theorem

Every set can be well ordered.

Trichonomy

Any two infinite cardinal numbers a and b are *comparable*. That is, at least one of the following holds: $a < b$, $a = b$, or $a > b$.

An Intricate Result

Let $\mathbb{R}$ denote the set of real numbers, $f : \mathbb{R} \to \mathbb{R}$, and $x_0 \in \mathbb{R}$. It is well-known that f is continuous at x_0 if and only if f is sequentially continuous at x_0. This result is usually taught in undergraduate analysis, and its proof makes an implicit use of the AC. In fact, assuming the validity of the AC is a necessary condition for this result to be true because as the following theorem shows, in ZF theory alone, sequential continuity does not imply continuity.

Theorem

Let $\mathbb{R}$ denote the set of real numbers and $x_0 \in \mathbb{R}$. In ZF theory alone, a function that is sequentially continuous at x_0 may not be continuous at x_0.

Sketch of the Proof

We show that it is possible to construct a function f which is sequentially continuous at x_0 but not continuous at x_0.

1. It can be shown that there is a model of ZF in which there exists some $X \subseteq \mathbb{R}$ such that X is both infinite and Dedekind finite.
2. Moreover, we can assume without loss of generality that X is bounded.
3. Then show that X has a limit point, say x_0.
4. If $x_0 \notin X$, then the characteristic function of X is not continuous at x_0, but it is sequentially continuous at x_0.
5. If $x_0 \in X$, then let $Y = X - \{x_0\}$. In this case, Y is an infinite, Dedekind finite, and bounded subset of $\mathbb{R}$. Hence, the previous point generates the desired function.

References

[1] Juliet Floyd and Akihiro Kanamori. How Gödel Transformed Set Theory. *Notices of the AMS*. **53**(4): 419–427, 2006.

[2] Leonard Gillman. Two Classical Surprises Concerning the Axiom of Choice and the Continuum Hypothesis. *The American Mathematical Monthly*. **106**(6): 544–553, 2002.

[3] Martin Aigner and Günter M. Ziegler. *Profs from The Book*. Springer-Verlag, Berlin, 2010.

[4] Paul R. Halmos. *Naive Set Theory*. Undergraduate Texts in Mathematics. Springer, New York, 1974.

Acknowledgments

I want to express my thanks to professor Dr. Condori for all of his support with the project.