Optimal Bioeconomics

A Multi-Species Fishery Model

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The Motivation

Our goal is to maximize the present value of a fishery which may harvest multiple interacting species. To do so, use the traditional present value integral

$$PV = \int_0^\infty R(t) \cdot e^{-\delta t} dt$$

Let q denote the catchability of a species in [vessel$\cdot$ day]$^{-1}$, E the effort in [vessels], and $h(t)$ the harvesting function

$$h(t) = qEx(t)$$

Along with the unit selling price p and cost per unit of effort c, the instantaneous net profit is

$$R(t) = ph(t) - cE$$

Writing the Present Value integral in terms of harvesting gives

$$PV = \int_0^\infty e^{-\delta t}(pqx(t)E - cE)dt$$

The question, then, is: How does one determine the optimal fishing effort so as to maximize value?

Population Dynamics with Harvesting

Before tackling the PV integral, one must determine the population dynamics for the species involved. The requisite dynamics will be constructed in stages:

First, recall the Logistic growth model for a species x:

$$\dot{x}(t) = x\left(r - \frac{r}{k}x\right)$$

where

- $x(t)$= the stock level at time t [tonnes]
- r= intrinsic growth rate [time]$^{-1}$
- k= carrying capacity [tonnes]

Letting subscripts denote each species and adding interaction terms gives rise to the familiar Lotka-Volterra predator-prey model

$$\begin{cases} \dot{x}_1(t) = & x_1(r_1 - \frac{r_1}{k_1}x_1 - \alpha_1 x_2) \\ \dot{x}_2(t) = & x_2(r_2 - \frac{r_2}{k_2}x_2 - \alpha_2 x_1) \end{cases}$$

Next, a stress term ϕA (predetermined and assumed constant) is added for each species. The net effect of ecological stress is to shift equilibrium points in the negative direction (which may result in overfishing if unaccounted for).

In the final stage, harvesting (recall $h(t) = qEx(t)$) is added to our system. The resulting population dynamics are then determined by the equations

$$\dot{x}_i = x_i\left(r_i - \sum_{j=1}^{n} \alpha_{ij}x_j - \phi_i A_i - q_i E\right); \;\; 1 \leqslant i \leqslant n$$

- A subscript i denotes a term for the i^{th} of n species
- α_{ij} is the interaction between the i^{th} and j^{th} species

Phase Planes of a Harvested 2x2 System

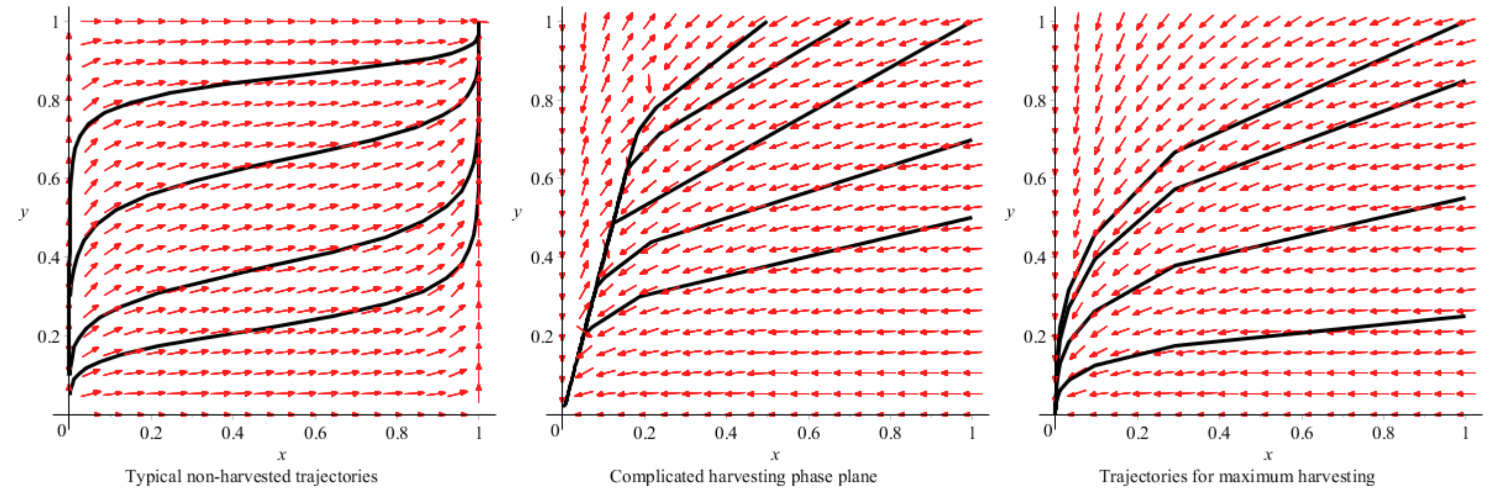

Figure: Sample trajectories for $u = 0$ (Left); $u = u_s$ (Middle); and $u = 1$ (Right)

Determining an Optimal Control Policy

The economic question may be formulated as an equivalent optimal control problem

$$\text{maximize } \mathcal{J}[\vec{x}(t), u(t)] = \int_0^\infty e^{-t}(\vec{\mu} \cdot \vec{x}(t) - 1)u(t)\, dt; \quad \vec{x}(t) = (x_1(t), ..., x_n(t))$$

$$\text{Subject to: } \dot{x}_i = x_i F_i(x_1, ..., x_n) - \gamma_i x_i u; \quad x_i(t) \in [0,1]; \quad u(t) \in [0,1]$$

Define the Hamiltonian as

$$\mathcal{H}(x_1, ..., x_n, \lambda_1, ..., \lambda_n, u, t) = ue^{-t}\left(\sum_{i=1}^{n}[\mu_i x_i] - 1\right) + \sum_{i=1}^{n} \lambda_i \dot{x}_i$$

Pontryagin's Maxmimum Principle states that the optimal trajectories x_i^* and control u^*, along with the corresponding Lagrange multipliers λ_i^*, maximize $\mathcal{H}$. Symbolically

- $$\mathcal{H}(x_1^*, ..., x_n^*, \lambda_1^*, ..., \lambda_n^*, u^*, t) \geqslant \mathcal{H}(x_1, ..., x_n, \lambda_1, ..., \lambda_n, u, t) \;\forall\, t \text{ and } \forall\, u \in [0,1]$$
- $$\frac{d\lambda_i}{dt} = -\frac{\partial H}{\partial x_i}$$

Substituting $\dot{x}_i$ into $\mathcal{H}$ and rewriting shows that the Hamiltonian is linear in the control u:

$$\mathcal{H}(...) = \sum_{i=1}^{n}[\lambda_i x_i F_i] + u \cdot \varphi; \qquad \varphi \overset{def}{=} \sum_{i=1}^{n}[e^{-t}\mu_i x_i - \gamma_i \lambda_i x_i - 1]$$

Since u^* must maximize $\mathcal{H}$, it is easy to see that we may define the control function implicitly as follows:

$$u^* = \begin{cases} 0 & \text{if } \varphi(t) < 0 \\ u_s(t) & \text{if } \varphi(t) \equiv 0 \\ 1 & \text{if } \varphi(t) > 0 \end{cases}$$

An explicit function for u^* is an arduous task but can be found by following the method of solution provided.

Method of Solution

To provide an answer to our economic problem we must first

- Determine necessary and sufficient conditions via the Maximum Principle
- Eliminate costate variables ($\lambda_i's$)
- Determine the switching function $u_s(t)$ for the singular path $\varphi \equiv 0$
  - The previous two steps make use of $\varphi' = \varphi'' = ... = \varphi^{(n)} = 0$
- Come to terms with the monster thus created

Harvesting a Predator-Prey System

To illustrate the policy, consider two species x and y with the following dynamics:

$$\begin{aligned} \dot{x} &= x\,(r_1 - \alpha_{11}x - \alpha_{12}y - \phi_1 A_1 - q_1 E) \\ \dot{y} &= y\,(r_2 + \alpha_{21}x - \alpha_{22}y - \phi_2 A_2 - q_2 E) \end{aligned}$$

Using the figure provided to analyze the phase planes, notice that

- Before harvesting (left) $\rightarrow u = 0$: Trajectories spawn from $(0,0)$ toward $(1,1)$
- Full harvesting (right) $\rightarrow u = 1$: Trajectories spawn from $(1,1)$ toward $(0,0)$
- The switching function u_s (middle) creates a 'region of equilibirum' (observe around (0.1,0.8)) above which neither species is driven toward extinction and the fishery is proiftable

Economic equilibria must be determined so that an optimal trajectory can be stitched together from each phase portrait. For example:

with initial stock levels of (0.05,0.1) the fishery must wait ($u = 0$) until the population of species two has increased sufficiently before applying an investment pulse ($u = 1$). Since maintaining maximum effort would lead to overfishing (and yield zero revenue), the switching function u_s must be used to attain an equilibrium that is ecologically and economically sound.

References and acknowgements

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